Numerical PDEs

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Advection-diffusion equations

as simple linear conservation laws

$u(x,t)$ in 1D

$u(\vec{r},t)$ in 2D/3D

is a scalar conserved quantity

(mass, energy, momentum)

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\Big[\underbrace{a(x,t)\, u(x,t)}_{\text{advective flux}}\Big] = \frac{\partial}{\partial x}\Big[\underbrace{d(x,t)\, u_x(x,t)}_{(-)\text{ diffusive/dissipative flux}}\Big]$$

advection velocity (given!)

diffusion coefficient (given!)

(entropy law) $d \geq 0$

1

$$\frac{\partial u(r,t)}{\partial t} + \nabla \cdot \underbrace{f(u,r)}_{} = 0$$

flux = advective + diffusive

In real world equations often non linear since advection velocity or diffusion coefficient depend on solution

$$u_t + (au)_x = (du_x)_x + f(u)$$

"reactions" sources/sinks

Higher dimensions

My notation:

$$\nabla = (\partial_x, \partial_y, \partial_z)^T = \text{grad}$$

$$\nabla \cdot \; = \text{div}$$

$$\nabla^2 = \nabla \cdot \nabla = \partial_{xx} + \partial_{yy} + \partial_{zz}$$

$$u_t + \nabla \cdot (\vec{a}\, u) = \nabla \cdot (D\, u)$$

velocity vector field

$D \in \mathbb{R}^{d \times d}$ $\succcurlyeq 0$

diffusion tensor

Einstein notation

$$\nabla \cdot (\vec{a} u) = \partial_\alpha (a_\alpha u) =$$

$$= (\partial_\alpha a_\alpha) u + a_\alpha (\partial_\alpha u)$$

$$= (\nabla \cdot \vec{a}) u + \vec{a} \cdot \nabla u$$

If velocity field is incompressible

$$\nabla \cdot \vec{a} = 0 \implies$$

$$\nabla \cdot (\vec{a} u) = \vec{a} \cdot \nabla u$$

but do not assume this

and don't use chain rule

but rather keep

advective flux $= \vec{a}\, u$

3

Why do we sometimes see advective derivative

$$D_t u = \partial_t u + a \cdot \nabla u$$

arise in fluid equations?

Either incompressible, or non-conservative /primitive form of equations (e.g., temperature instead of energy)

E.g. $c = \frac{u}{\rho}$ $u \leftarrow$ solute density, $\rho \leftarrow$ density, c ↑ concentration

$$u = \rho c \Rightarrow u_t = \rho c_t + c \rho_t$$

$$\rho_t = -\nabla \cdot (\vec{a} \rho) \quad (\underline{no} \text{ diffusion of mass})$$

$$u_t + \nabla\cdot(\vec{a}u) =$$

$$u_t + \nabla\cdot(\rho c \vec{a}) =$$

$$\underbrace{\rho c_t - c\,\cancel{\nabla\cdot(\vec{a}\rho)}} + \underbrace{c\,\cancel{\nabla\cdot(\vec{a}\rho)} + \rho\vec{a}\cdot\nabla c}$$

$$= \rho\left(c_t + \vec{a}\cdot\nabla c\right) = \rho D_t c$$

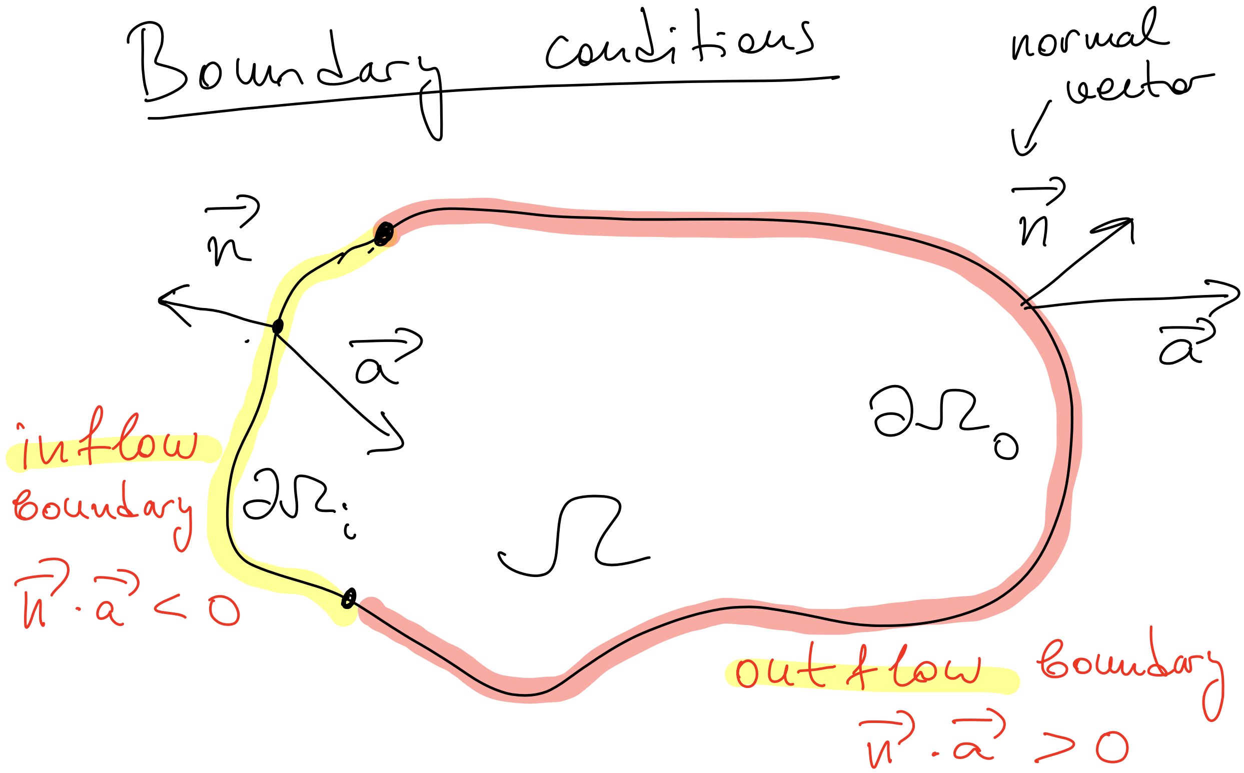
$$\Rightarrow \quad \rho D_t c = \nabla\cdot(\text{diffusive flux})$$

$$= \nabla\cdot(\underbrace{\rho D \nabla c}_{\text{Fick's diffusion law}})$$

$$\rho D_t c = \nabla\cdot(\rho D \nabla c)$$

(not conservative)
unless $\rho = \text{const}$

Boundary conditions



Information comes into the domain at inflow boundary & flows outside of domain at outflow boundary.

For advection equation:

Dirichlet BC on $\partial\Omega_{inflow}$

No BC on $\partial\Omega_{outflow}$

For adv-diff equation, if
$D > 0$ everywhere
Need flux BC everywhere
on all of $\partial\Omega$

"Neumann" / Robin BC:

given $\vec{n} \cdot \vec{f} = \vec{n} \cdot (\vec{a} u - D \nabla u)$

OR

Dirichlet BC:

given u itself
on pieces of $\partial\Omega$

We see that there is a change in character of PDE as diffusion becomes weaker compared to advection.

Before solving equation, we must know whether it is advection-dominated or diffusion-dominated

This is a property of the problem (the PDE + parameters)

If characteristic length scale of physical problem is L, and characteristic speed $\sim \|\vec{a}\|$ is V, and typical diffusion coefficient is D, then

$$\vec{a} \cdot \nabla u \sim V \frac{U}{L}$$

$$\nabla \cdot (D \nabla u) \sim D \frac{U}{L^2}$$

Péchlet number

$$P_e = \frac{\text{advection}}{\text{diffusion}} = \frac{VL}{D}$$

If $P_e >> 1$ problem is advection-dominated & we should use methods developed for advection equation (pure hyperbolic eqs.)

For numerical methods, the important length scale is the grid size h

$$\text{Cell } Pe = \frac{Vh}{D}$$ (compare to 1)

If grid is very fine, $h << L$ & problem is resolved.