For linear problems all these methods work equally well. However, for nonlinear ones (and in the presence of variable coefficients) the pseudo-spectral (collocation) approach is particularly easy to apply since it involves the products of numbers (solution/variable coefficient's values in collocation points) instead of products of expansions, which are much more difficult to handle.

The convergence of pseudo-spectral approximations for very smooth functions is always geometrical, *i.e.* $\sim \mathcal{O}(q^N)$, where N is the number of modes. This statement is true for any derivative with the same convergence factor $0 < q < 1$. However, the periodic pseudo-spectral method converges always faster than its non-periodic counterpart. This conclusion follows from convergence properties of Fourier and Tchebyshev series in the complex domain.

The relative resolution ability of various pseudo-spectral methods can be also quantified in terms of the number of points per wavelength needed to resolve a signal. Indeed, this description is more suitable for wave propagation problems. For periodic Fourier-type methods one needs 2 points per wavelength. For Tchebyshev-type methods one needs about π points. Finally, this number goes up to 6 nodes per wavelength for uniform grids. However, due to the huge Lebesgue constant Λ_N^{Uni} the uniform grids in pseudo-spectral setting are not usable in practice.

3. Aliasing, interpolation and truncation

Let us take a continuous (and possibly a smooth) function $u(x)$ defined on the interval $\mathcal{I} = (-\pi, \pi)$ and develop it in a Fourier series. In general it will contain the whole spectrum (*i.e.* the infinite number) of frequencies:

$$u(x) = \sum_{k=-\infty}^{+\infty} v_k \, \mathrm{e}^{\mathrm{i}kx}.$$

Now let us discretize the interval $\mathcal{I}$ with N equispaced collocation points (as we do it in Fourier-type pseudo-spectral methods. In the following we assume N to be odd, *i.e.* $N = 2m+1$. On this discrete grid all modes $\left\{\mathrm{e}^{\mathrm{i}(k+jN)x}\right\}_{j\in\mathbb{Z}}$ are indistinguishable. See Figure 4 for an illustration of this phenomenon.

The interpolating trigonometric polynomial on a given grid can be written as

$$\mathbb{I}_N[u] = \sum_{k=-m}^{m} \hat{v}_k \, \mathrm{e}^{\mathrm{i}kx}.$$

Each discrete Fourier coefficient incorporates the contributions of all modes which looks the same on the considered grid:

$$\hat{v}_k = \sum_{j=-\infty}^{+\infty} v_{k+jN}.$$

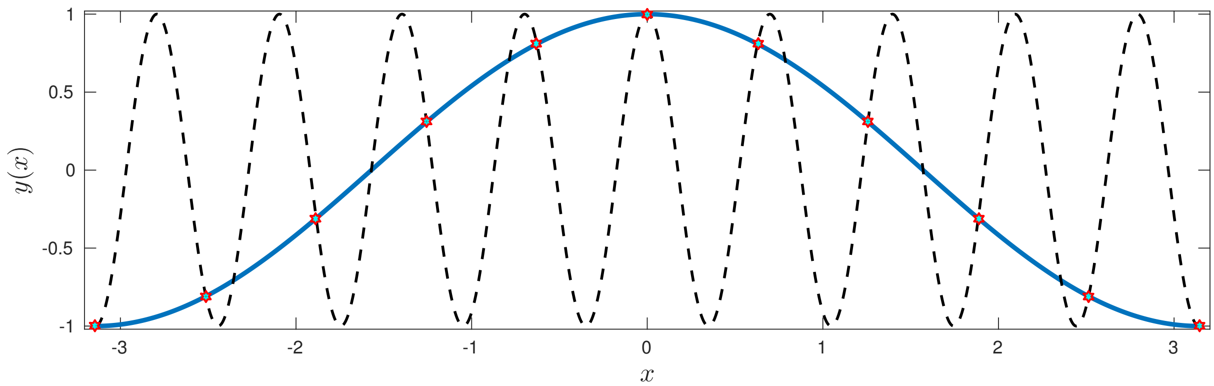

Figure 4. *Illustration of the aliasing phenomenon: two* Fourier *modes are indistinguishable on the discrete grid. The modes represented here are* $\cos(x)$ *and* $\cos(9x)$ *and the discrete grid is composed of* $N = 11$ *equispaced points on the segment* $[-\pi, \pi]$.

Let us recall that the polynomial $\mathbb{I}_N[u]$ takes the prescribed values $\{u(x_k)\}_{k=-m}^{m}$ in the points of the grid $\{x_k\}_{k=-m}^{m}$. This object is fundamentally different from the truncated Fourier series:

$$\mathbb{T}_N[u] = \sum_{k=-m}^{m} v_k \, \mathrm{e}^{\mathrm{i}kx}.$$

The difference between these two quantities is known as the *aliasing error*:

$$\mathscr{R}_N[u] \stackrel{\mathrm{def}}{:=} \mathbb{I}_N[u] - \mathbb{T}_N[u] = \sum_{k=-m}^{m} \sum_{\forall j \neq 0} v_{k+jN} \, \mathrm{e}^{\mathrm{i}kx}.$$

After applying the Pythagoras theorem*, we obtain

$$\| u - \mathbb{I}_N[u] \|_{L_2}^2 = \| u - \mathbb{T}_N[u] \|_{L_2}^2 + \| \mathscr{R}_N[u] \|_{L_2}^2.$$

Thus, the interpolation error is always larger than the truncation error in the standard L_2 norm. The amount of this difference is precisely equal to the committed *aliasing error*. However, we prefer to use in pseudo-spectral methods the interpolation technique because of the Discrete Fourier Transform (DFT), which allows to transform quickly (thanks to the FFT algorithm) from the set of function values in grid points to the set of its interpolation coefficients. So, it is easier to apply an FFT instead of computing N integrals to determine the Fourier series coefficients.

*We can apply the Pythagoras theorem since the aliasing error $\mathscr{R}_N[u]$ contains the Fourier modes with numbers $|k| \leqslant m$, while the reminder $u - \mathbb{T}_N[u]$ contains only the modes with $|k| > m$. Thus, they are orthogonal.

Nonlinearities. Let us take the simplest possible nonlinearity — the product of two functions $u(x)$ and $v(x)$ defined by their truncated FOURIER series containing the modes up to m:

$$u(x) = \sum_{k=-m}^{m} u_k\, \mathrm{e}^{ikx}, \qquad v(x) = \sum_{k=-m}^{m} v_k\, \mathrm{e}^{ikx}.$$

The product of these two functions $w(x)$ can be obtained my multiplying the FOURIER series:

$$w(x) = u(x)\cdot v(x) = \Bigl(\sum_{k=-m}^{m} u_k\, \mathrm{e}^{ikx}\Bigr)\cdot\Bigl(\sum_{k=-m}^{m} v_k\, \mathrm{e}^{ikx}\Bigr) \equiv \sum_{k=-2m}^{2m} w_k\, \mathrm{e}^{ikx}.$$

It can be clearly seen that the product contains high order harmonics up to $\mathrm{e}^{\pm imx}$ which cannot be represented on the initial grid. Thus, they will contribute to the aliasing error explained above.

The aliasing of a nonlinear product can be ingeniously avoided by adopting the so-called $3/2^{\mathrm{th}}$ rule whose MATLAB implementation is given below. This function assumes that input vectors are FOURIER coefficients of functions $u(x)$ and $v(x)$. The resulting vector contains (anti-aliased) FOURIER coefficients of their product $w(x) = u(x)\cdot v(x)$.

```
function w_hat = AntiAlias(u_hat, v_hat)
  N         = length(u_hat);
  M         = 3*N/2; % 3/2th rule
  u_hat_pad = [u_hat(1:N/2) zeros(1, M-N) u_hat(N/2+1:end)];
  v_hat_pad = [v_hat(1:N/2) zeros(1, M-N) v_hat(N/2+1:end)];
  u_pad     = ifft(u_hat_pad);
  v_pad     = ifft(v_hat_pad);
  w_pad     = u_pad.*v_pad;
  w_pad_hat = fft(w_pad);
  w_hat     = 3/2*[w_pad_hat(1:N/2) w_pad_hat(M-N/2+1:M)];
end % AntiAlias()
```

The main idea behind is to complete vectors of FOURIER coefficients by a sufficient number of zeros (*i.e.* the so-called zero padding technique) so that in the physical space the product $u(x)\cdot v(x)$ can be fully resolved. The final step consists in extracting m relevant FOURIER coefficients [35, 37].

Remark 7. *To Author's knowledge, the development of efficient and rigorously justified anti-aliasing rules for other types of nonlinearities such as the division, square root,* etc. *is an open problem.*

3.1. Example of a second order boundary value problem

Consider the following second order Boundary Value Problem (BVP) on the interval $\mathcal{I} = [-1, 1]$:

$$u_{xx} + u_x - 2u + 2 = 0, \qquad u(-1) = u(1) = 0. \tag{3.1}$$