Nonlinear Conservation Laws: Godunov method

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Let's consider the nonlinear conservation law

$$q_t + \left(f(q)\right)_x = 0$$ new notation

$f(q)$: flux function

where q does not have to be scalar. Famous examples include:

a) Burger's equation $f(q) = \frac{q^2}{2}$

b) Euler's equations of inviscid fluid dynamics (aerodynamics)

c) Shallow water equations (geofluids)

To ensure consistency with weak form of PDE it is crucial to use a conservative FV method (Lax–Wendroff theorem):

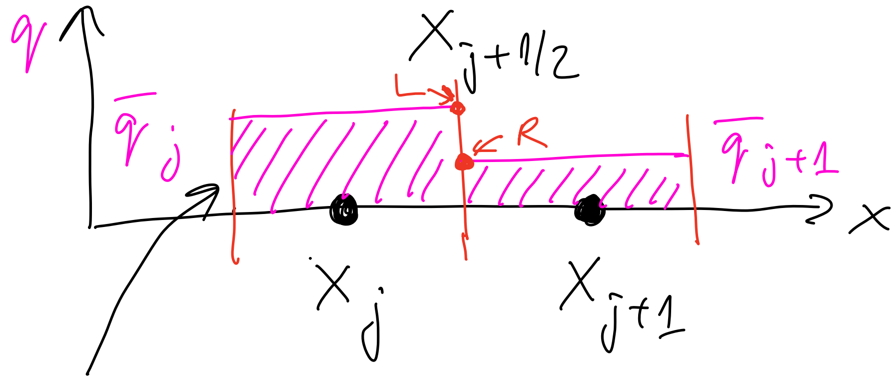
$$\bar{q}_i^{n+1} = \bar{q}_i^{n} - \frac{\Delta t}{\Delta x}\left[f_{i+1/2}^{n+1/2}(\bar{q}) - f_{i-1/2}^{n+1/2}(\bar{q}) \right]$$

where

$$f_{i+1/2}^{n+1/2}(\bar{q}) \approx \frac{1}{\Delta t}\int_{t^n}^{t^n+\Delta t} f(q(x_{i+1/2}, t))\,dt$$

What is the equivalent of upwind & the 2nd order methods we did for advection for this more general case?

For upwind we used a

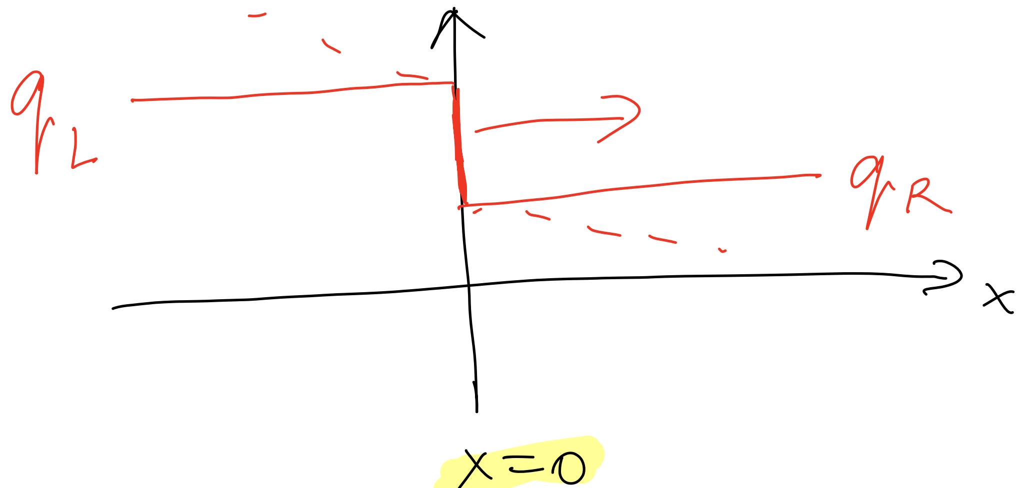


Piecewise constant reconstruction

$$q(x_{j-1/2} < x < x_{j+1/2}) = \bar{q}_j + O(h^2)$$

At every face $x_{i+1/2}$ we need to compute a flux. If we satisfy a Courant condition: characteristics emanating from cell faces at t_n don't reach other faces until time t_{n+1}, then we can focus on a single face to estimate flux

At each interface we have a so-called Riemann Problem



$$q(x, t=0) = \begin{cases} q_L & \text{if } x<0 \\ q_R & \text{if } x>0 \end{cases}$$

Observe that this IC has no intrinsic length scale: If we rescale space by a factor and also rescale time by the same factor, we get the same solution!

$\Rightarrow \quad q(x,t) = q(x/t)$ for Riemann problem

This means in particular that

$$q(x=0,t) = q(x/t=0) = \text{const}$$

$$q_{\text{Riemann}}(x=0,t) = q^{\downarrow}(q_L, q_R)$$

for $t_n \leq t < t_{n+1}$

And therefore the flux accross the interface $x=0$ is <u>constant</u> for $t_n \leq t < t_{n+1}$ and therefore trivial to integrate over the time step!

Note: For advection

$$q^{\downarrow} = \begin{cases} q_L & \text{if } a>0 \\ q_R & \text{if } a<0 \end{cases}$$

$$f_{i+1/2}^{n+1/2} = f\left(q^{\downarrow}\left(\bar{q}_i, \bar{q}_{i+1}\right)\right)$$

comes from

$\boxed{\text{Riemann solver}}$

Godunov flux

(or Godunov method)

We see that for linear advection Godunov's method is simple upwinding, so it is the natural generalization of upwinding to nonlinear conservation laws. It is only first-order accurate but typically the most robust method, i.e. in most cases it converges to a suitable solution of the PDE, though this is HARD to prove.

All of the complexity lies in the Riemann solver, and that requires a lot of hyperbolic PDE theory, which we won't go into here. In fact, the Riemann problem has been solved exactly only for a few equations in 1D and it can get very complicated. In practice, codes typically use approximate Riemann solvers, e.g., based on linearizing the PDE (can fail for certain types of solutions).
Here, for brevity, we will illustrate on simple 1D PDE and assume a Riemann solver exists otherwise

Burgers equation

or similar scalar

$$q_t + (f(q))_x = q_t + \left(\frac{q^2}{2}\right)_x = 0$$

Do **NOT** use the chain rule

$$\cancel{q_t + f'(q)\, q_x = 0}$$

since this destroys weak formulation.

Assume **convex** (or concave) flux, $f''(q)$ has same sign over range of q's.

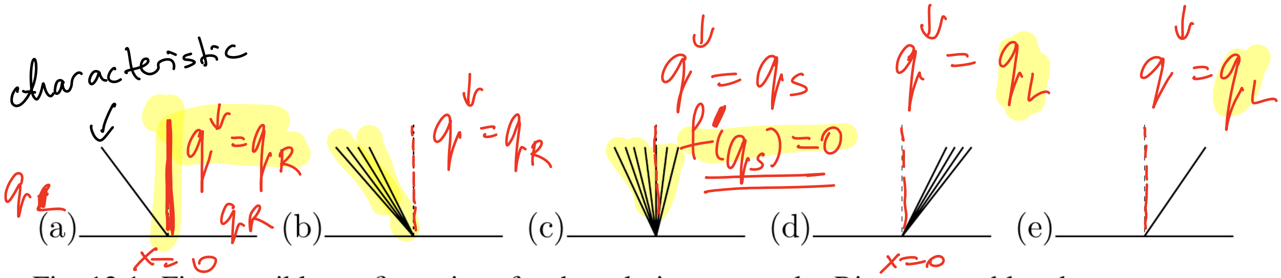


Fig. 12.1. Five possible configurations for the solution to a scalar Riemann problem between states Q_{i-1} and Q_i, shown in the x–t plane: (a) left-going shock, $Q^{\downarrow}_{i-1/2} = Q_i$; (b) left-going rarefaction, $Q^{\downarrow}_{i-1/2} = Q_i$; (c) transonic rarefaction, $Q^{\downarrow}_{i-1/2} = q_s$; (d) right-going rarefaction, $Q^{\downarrow}_{i-1/2} = Q_{i-1}$; (e) right-going shock, $Q^{\downarrow}_{i-1/2} = Q_{i-1}$.

$$q^{\downarrow} = \begin{cases} q_L \text{ or } q_R \\ \text{or} \\ 0 \text{ for Burgers} \end{cases}$$

Reminder: For Burgers,

$$q_t + q(q_x) = 0$$

(when smooth) so slope of straight characteristics is q, i.e. q_L for left characteristics, q_R for right characteristics.

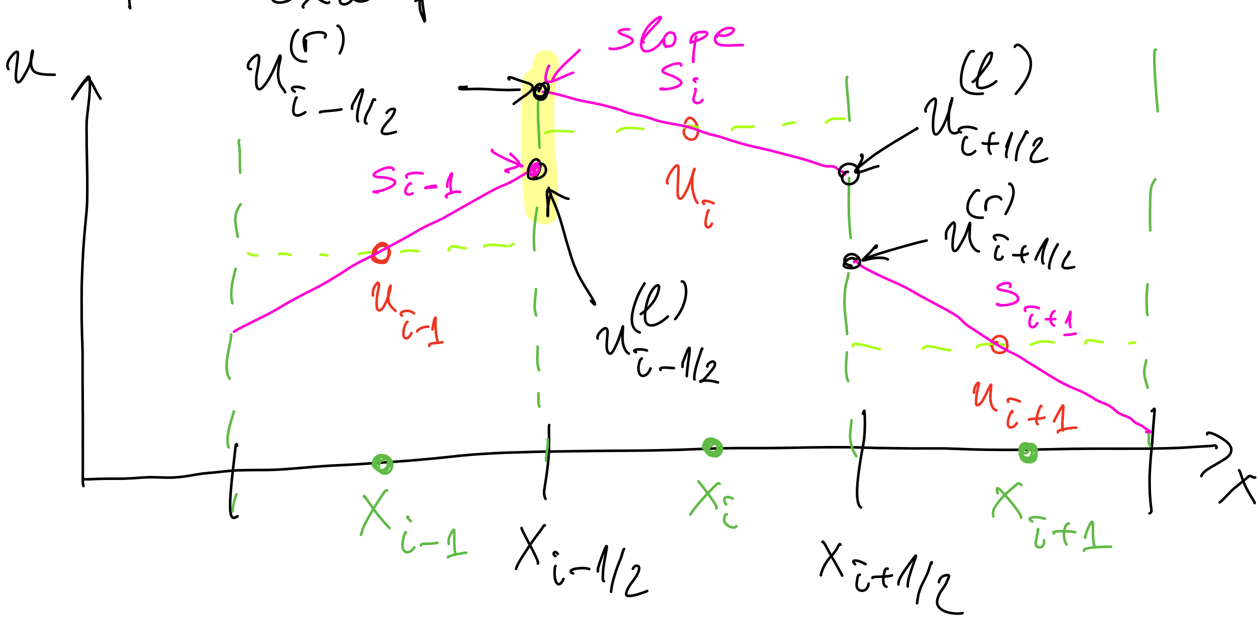
Turns out for convex flux

$$f_{j+1/2}^{n+1/2} = \begin{cases} \min\limits_{\bar{q}_j^n \le \theta \le \bar{q}_{j+1}^n} f(\theta) & \text{if } \bar{q}_j^n < q_{j+1}^n \\ \text{or} & \\ \max\limits_{\bar{q}_{j+1} \le \theta \le \bar{q}_j^n} f(\theta) & \text{otherwise} \end{cases}$$

For Burgers, we get $f_{j+1/2}^{n+1/2}$ is either $q_j^2/2$, $q_{j+1}^2/2$, or $\emptyset$.

How do we get 2nd order accuracy? I will only show one method (not guaranteed to converge but often works OK) that generalizes what we did for advection.

Recall for advection we went to **higher-order reconstruction**, for example linear:



[Or <u>quadratic</u> reconstruction for MOL 3rd order.]

$$q(x_{i-1/2} < x < x_{i+1/2}) = \bar{q}_i^n \leftarrow \text{conservation}$$

$$+ s_i^n (x - x_i) \quad + O(h^2)$$

slope, potentially <u>limited</u> using something like MC limiter.

If q is a vector, do separately for each component (this makes the method not quite consistent with characteristics but makes it simple)

This gives at each face:

$$\begin{cases} q_{i+1/2,L}^n = \bar{q}_i^n + \frac{\Delta x}{2} S_i \\ q_{i+1/2,R}^n = \bar{q}_{i+1}^n - \frac{\Delta x}{2} S_{i+1} \end{cases}$$

Now, we also want to estimate state at midpoint in time to get 2^{nd} order accuracy, using Taylor series

$$\begin{cases} q^{n+1/2}_{i+1/2,L} = \bar{q}^{\,n}_{i+1/2,L} + \dfrac{\Delta t}{2}\left(u_t\right)^n_i \\[2ex] q^{n+1/2}_{i+1/2,R} = \bar{q}^{\,n}_{i+1/2,R} + \dfrac{\Delta t}{2}\left(u_t\right)^n_{i+1} \end{cases}$$

where time derivative estimate

$$f(u(x))_x = \frac{\partial f}{\partial u} u_x$$

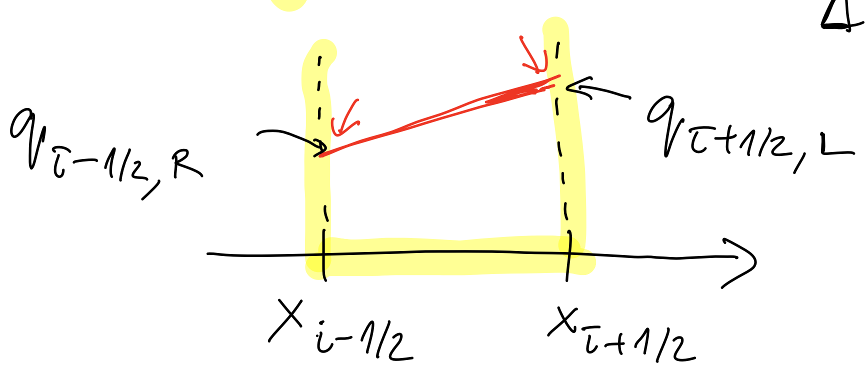
Fix

$$\rightarrow \left(u_t\right)^n_i = - \frac{f\left(q^n_{i+1/2,L}\right) - f\left(q_{i-1/2,R}\right)}{\Delta x}$$

Or, use

$$\left(u_t\right)^n_i = -\left(\frac{\partial f}{\partial u}\right)^n_i S_i$$

analytical Jacobian



Note that limiting and extrapolation to faces at midpoint in time (skip this for MOL schemes) is done cell by cell, once initial slope are estimated (centered ala Fromm):

$$\left(S_i^n\right)_{\substack{\text{not}\\ \text{limited}}} = \frac{\bar{q}_{i+1}^{\,n} - \bar{q}_{i-1}^{\,n}}{2\Delta x}$$

Now, we use the Riemann solver

$$f_{i+1/2}^{n+1/2} = f\left(q\left(q_{i+1/2,L}^{n+1/2},\ q_{i+1/2,R}^{n+1/2}\right)\right)$$

Standard Riemann solver for piecewise constant initial data, not necessary to solve Riemann problem for linear reconstruction

This kind of scheme is called MUSCL-Hancock in some sources. For advection, assuming $a(x,t) > 0$ in the domain, it gives: $\quad u_t + (a(x)u)_x = 0$

$$f^{n+1/2}_{i+1/2} = a_{i+1/2} \, u^{n+1/2}_{i+1/2,L}$$

$$u^{n+1/2}_{i+1/2,L} = \bar{u}_i + \frac{\Delta x}{2} s_i - \frac{\Delta t}{2\Delta x}\Big[\underbrace{a_{i+1/2} u_{i+1/2,L} - a_{i-1/2} u_{i-1/2,R}}_{-\Delta x (u_t)_i} \Big]$$

Let $a_{i-1/2} = a_{i+1/2} - \Delta a_i$

$$\Rightarrow -(u_t)_i \Delta x = a_{i+1/2}\left(\bar{u}_i + \frac{\Delta x}{2} s_i\right) - a_{i+1/2}\left(\bar{u}_i - \frac{\Delta x}{2} s_i\right) + \Delta a_i\left(\bar{u}_i - \frac{\Delta x}{2} s_i\right)$$

$$= \Delta x \cdot S_i \underbrace{\left(a_{i+1/2} - \frac{\Delta a_i}{2} \right)}_{(a_{i+1/2} + a_{i-1/2})/2} + \Delta a_i \bar{u}_i \quad \Rightarrow$$

$$u_{i+1/2}^{n+1/2} = \bar{u}_i + \frac{\Delta x}{2} S_i - \frac{\Delta t}{2} \left(\underbrace{\frac{a_{i+1/2} + a_{i-1/2}}{2} S_i}_{\text{I put } a_{i+1/2} \text{ here earlier}} + \frac{\overbrace{a_{i+1/2} - a_{i-1/2}}^{\text{used chain rule earlier}}}{\Delta x} u_i \right)$$

or put a_i

Fix