Finite Element Methods

A. Donev, Fall 2021

These lecture notes are based on lecture notes by Georg Stadler. I consider the background on piecewise smooth local basis functions to be part of interpolation in 2D/3D so see those lecture notes first.

Like pseudo spectral methods, FEM is a series method, meaning that the discrete solution is a function that is a sum of basis functions and the discrete unknowns are the series coefficients:

$$u_h(x) = \sum_{i=1}^{N} U_i \varphi_i(x) \approx u(x)$$

"grid size" (discrete) — h

unknown coefficients — U_i

basis functions — φ_i

A key difference is that now the basis functions $\varphi_i(x)$ are piecewise polynomials with localized support — this will be key for efficiency as it will lead to sparse matrices not dense like for orthogonal polynomials.

But the heart of FEM methods is their relation to weak & variational formulation of elliptic (parabolic) PDEs

Consider PDE on bounded domain $\Omega \in \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$:

$\partial_j \equiv \partial/\partial x_j$

$$-\sum_{i,j=1}^{n} \underbrace{\partial_j \left(a_{ij}(x)\, \partial_i u\right)}_{\text{diffusion}} \equiv -\nabla\cdot\left(A(x)\nabla u\right)$$

$$+ \sum_{i=1}^{n} \underbrace{b_i(x)\, \partial_i u}_{\text{advection}} \equiv b(x)\cdot\nabla u$$

$$+ \; c(x)\, u = f(x)$$

Importantly, $A(x)$ is uniformly (symmetric) positive definite, i.e., $\nabla\cdot(A\nabla u)$ is an elliptic operator.

BCs can be Dirichlet (u), Neumann ($\partial u/\partial n$), or Robin/mixed.

If $b = 0$ (no "advection"),
we have a variational
formulation of PDE. Take
for simplicity

$$v(x)\begin{cases} -\nabla\cdot(A\nabla u) + u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega_1 \quad \text{(essential BC)} \\ a\dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega_2 \quad \text{(natural BC)} \end{cases}$$

Take a test function $v \in C^1(\Omega)$
with $v|_{\partial\Omega_1} = 0$ (essential BCs
must be incorporated into FEM
spaces / enforced explicitly in
the strong sense), multiply
PDE and integrate by parts
to lower smoothness requirements

$$-\int_\Omega \nabla\cdot(a\nabla u)\, v\, dx + \int_\Omega u v\, dx =$$

$= g$ on $\partial\Omega_2$

$$= \int_\Omega a \nabla u\cdot\nabla v\, dx - \int_{\partial\Omega} a\frac{\partial u}{\partial n}\, v\, ds$$

zero on $\partial\Omega_1$

$$+ \int_\Omega u v\, dx = \int_\Omega f v\, dx$$

Using BCs we get

$$\underbrace{\int_\Omega a \nabla u\cdot\nabla v\, dx + \int_\Omega u v\, dx}_{=a(u,v)} = \underbrace{\int_\Omega f v\, dx + \int_{\partial\Omega_2} g v\, dx}_{\ell(v)} \quad \ldots (*)$$

Weak formulation: $(*)$ is true for all suitable $v(x)$

The right function space is the same for u and v (for self-adjoint problems, but in Petrov-Galerkin methods u and v belong to <u>different</u> spaces) is

$$H^1_{0,\partial\Omega_1}(\Omega) = \Big\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \ \forall i = 1, \dots, n, \quad u = 0 \text{ on } \partial\Omega_1 \Big\}$$

Sobolev space $\quad H^1 \equiv W^{1,2}$

Denote bilinear form

$$a(\underset{\underset{H^1_0}{\cap}}{u}, v) = \int_\Omega a(x) \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx$$

and linear form

$$\ell(v) = \int_{\Omega} f v \, dx + \int_{\partial\Omega_2} g v \, ds$$

Variational / weak form of PDE:

$$\boxed{\begin{cases} \underline{a(u,v)} = \underline{\ell(v)}, \quad u \in H^1_{0,\partial\Omega_1} \\ \forall v \in H^1_{0,\partial\Omega_1} \end{cases}}$$

If $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega_2)$

then Lax-Milgram lemma

says u is a unique solution.

Key condition is coercivity/ellipticity:

$$a(v,v) \geq c_0 \|v\|_{H^1}$$

$$(v,w)_{H^1} = \int_{\Omega} (\nabla v \cdot \nabla w + v \cdot w) \, dx$$

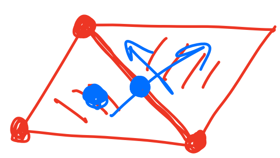
If $A(x)$ is SPD for all x,

$$a(u,v) = a(v,u)$$

we have also equivalent energy/variational formulation

$$u = \arg\min_{v \in H^1_{0,\partial\Omega_1}} J(v)$$

$$J(v) = \frac{1}{2} a(v,v) - l(v)$$



Steps in FEM

1) Write weak form of PDE (calculus)
2) Choose finite dimensional spaces for all function spaces (theory)
3) Solve resulting system of equations (practical)

So instead of

Find $u \in V$ s.t. $a(u, v) = \ell(v)\ \forall v$

choose finite-dimensional $V_h \subset V$ made of piecewise polynomial functions and solve

$$\boxed{\text{Find } u_h \in V_h \text{ s.t. } a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h}$$

Discrete weak form

by solving a system of equations.

$$V_h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_N\}$$

(linearly independent but not orthogonal)

$$u_h = \sum_{i=1}^{N} U_i \varphi_i(x)$$

Plug into weak form to get

Take $v_h = \varphi_j$ ←

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$$\sum_{i=1}^{N} a(\varphi_i, \varphi_j)\, U_i = \ell(\varphi_j) \quad \forall j = 1, \dots, N$$

$$\begin{cases} AU = L & \text{— system of } N \text{ equations} \\ A_{ij} = a(\varphi_i, \varphi_j) & \text{stiffness matrix} \end{cases}$$

$$L_i = \ell(\varphi_i)$$

$$\int_\Omega a(x) \nabla\varphi_i \nabla\varphi_j \, dx$$

Notes:

① Since computing A_{ij} requires integration, it may have to itself be approximated by spectral quadrature (e.g. Gauss quad). Always true for r.h.s. L

② By choosing piecewise basis wisely we can make A be sparse & SPD and thus solve system more efficiently

Parabolic problems (aside)

$$\begin{cases} \dfrac{\partial u}{\partial t} + a u_x = d u_{xx} + b u + f(x,t) \\ u(\partial\Omega) = 0 \end{cases}$$

constant coeff.

Method of lines:

$$u(x,t) = (u(t))(x)$$

$$u : (0,T) \to H^1_0(\Omega)$$

Weak form:

$$\int_\Omega \overset{\varphi_j}{v} \overset{\varphi_i}{u_t}\, dx = \int_\Omega v(-a u_x + d u_{xx} + b u + f) \quad \forall v \in H^1_0(\Omega)$$

integrate by parts

$$u(x,t) = \sum_{i=1}^{N} U_i(t) \varphi_i(x)$$

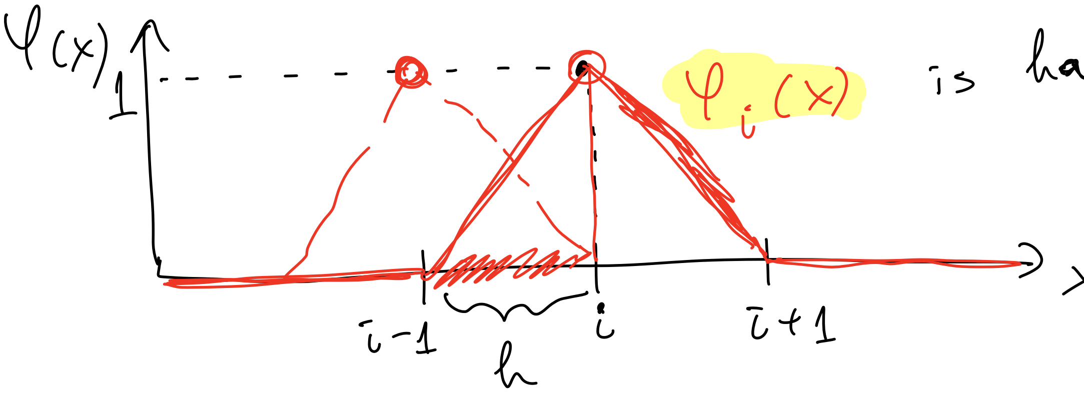
gives

$$M \frac{dU}{dt} + \boxed{AU = g} \quad \text{(ODEs)}$$

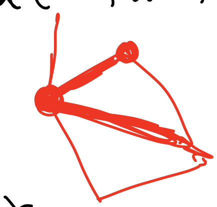
mass matrix

$$M_{ij} = (\varphi_j, \varphi_i)_{L^2(\Omega)}$$

Take uniform grid in 1D



is hat function



$$\Rightarrow \int \varphi_j \varphi_{j+1} \, dx = \frac{h}{6}$$

$$\int \varphi_j^2 \, dx = \frac{2}{3} h$$

$$- \int \frac{d\varphi_{j-1}}{dx} \varphi_j \, dx = 1/2$$

$$\int \left(\frac{d\varphi_j}{dx}\right)^2 dx = \frac{2}{h^2}$$

$$\int \left(\frac{d\varphi_{j-1}}{dx}\right)\left(\frac{d\varphi_j}{dx}\right) dx = -\frac{1}{h^2}$$

$\int \varphi_i \varphi_j \, dx$

$i = j$

or $|i-j| = 1$

Gives discretization

$$M \frac{dU}{dt} + a\tilde{D}U = d D_2 U + b M U + \mathbf{F}$$

where $M = \frac{1}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & 1 \\ & \ddots & \ddots & \\ & & 1 & 4 \end{bmatrix}$ mass matrix

$$\tilde{D} = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix} = \text{centered difference}$$

$$D_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} = \text{standard Laplacian}$$

$$F_i = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} f(x)\,\varphi_i(x)\,dx$$

Except for mass matrix, this is the same as the FD second order!

We know that centered difference is not good for advection (will require RK3+ to integrate).

But FEM can be higher order (with some conditioning issues) & use unstructured grids.

Note that

$$M^{-1}\tilde{D}\,U \simeq \frac{\partial u}{\partial x} + O(h^4)$$

in the finite difference sense (called "compact finite difference")

So in practice the method will be better than 2^{nd} order FD for advection. But each timestep requires solving $Mx=b$!

Lumped mass approximation: Approx. M by a diagonal matrix

Back to time-independent problems

We will not go into the extensive & well-developed theory of FEM methods, which relies heavily on Sobolev function spaces. Some notes:

(1) Cea Lemma:

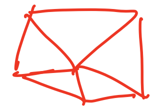
The FEM solution is nearly optimal in the approximation space:

$$\|u-u_h\|_{H^1} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u-v_h\|_{H^1}$$

$u_h \in V_h$

As long as the constants c_1 and c_0 are well-behaved, and the approximation is suited to the PDE, we don't have to worry & have strong theoretical guarantees.

(2) The target in the FEM world is to prove a priori error bound



$$\|u-u_h\|_{H^1} \le C h^{p} \quad \leftarrow \text{grid size}$$

where p is the degree of the polynomial basis functions (so linear gives first order convergence in H^1 in general)

(3) For purely elliptic PDEs, define inner product

$$(v,w)_a = a(v,w)$$

From

$$\begin{aligned}
&\text{PDE} \quad a(u, v_h) = \ell(v_h) \qquad \forall v_h \in V_h \\
&\text{FEM} \quad a(u_h, v_h) = \ell(v_h) \\
&\Rightarrow a(\underbrace{u-u_h}_{\text{error}}, v_h) = 0 \qquad (\varepsilon, v_h)_a = 0
\end{aligned}$$

$\Rightarrow$ Error is orthogonal to V_h in the new inner product, i.e.

$$\|u-u_h\|_a = \min_{v_h \in V_h} \|u-v_h\|_a$$

FEM approximation is optimal in the a-norm (improved Cea lemma)

For example, for

$$\begin{cases} -u'' + u = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

and a regular 1D grid using:

Cea's lemma + interpolation error bound + elliptic regularity ($\|u\|_{H_2} \leq C \|f\|_{L_2}$) one gets:

$$\|u-u_h\|_{H^1} \leq \frac{2h}{\pi}\left(1+\frac{h^2}{\pi^2}\right)^{1/2} \|f\|_{L_2}$$

$$\|u-u_h\|_{L_2} \leq \frac{2h^2}{\pi}\left(1+\frac{h^2}{\pi^2}\right)^{1/2} \|f\|_{L_2}$$

piecewise linear elements

However, since we know that for regular grids + linear basis FEM is the same as FD 2^{nd} order, we expect that the solution is more accurate than just 1^{st} order. For this one needs to switch to a different norm that does not test derivatives since those are indeed only first-order accurate. Specifically, one can show

$$\| u - u_h \|_{L_2} \leq 4 h^2 |u|_{H^2} \leq O(h^2) \|f\|_{L_2}$$

i.e. solution is second-order accurate in L_2 norm.

However, FEM error bounds can become useless if the approximation space is not suited to the PDE.
Notably, for advection-diffusion:

$$\begin{cases} -d\,\nabla^2 u + \vec{a}\cdot\nabla u = f \\ \nabla\cdot\vec{a} = 0 \end{cases}$$

the standard FEM discretization gives an error constant

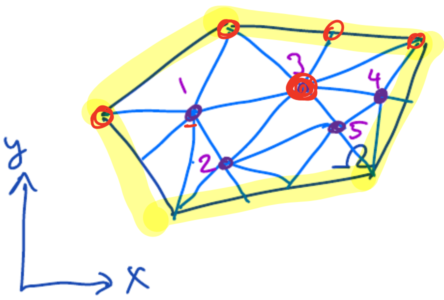
$$C \sim \sqrt{1+P_e^2}$$

where P_e is the Pechlet number. So for advection-dominated problems $C \gg 1$ and FEM does not work well without some "stabilization"

Some practicalities:

FEM Grids & Matrices

$\Omega \subset \mathbb{R}^2$ polygonial boundary, Cover Ω with triangles



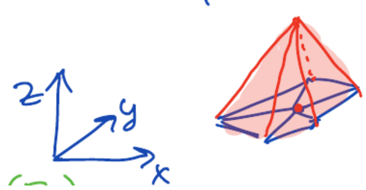
• interior point

V_h ... space of continuous functions that are linear on each triangle

$V_h \subset V$

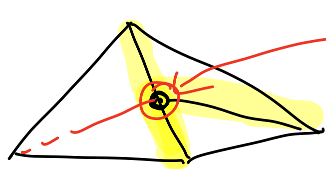
ϕ_i basis for each interior node, $i = 1, \dots, 5$

2-dim. hat functions:



$\phi_i(x,y)$

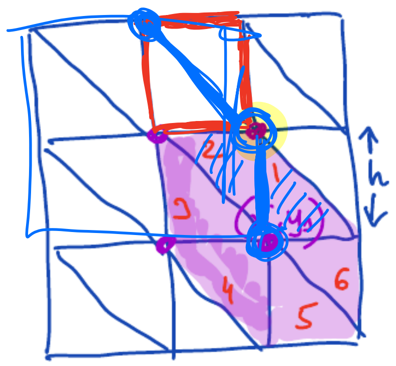
In 2D, almost any domain of interest can be triangulated, so take FEM cells to be triangles, FEM nodes to be the vertices, no hanging nodes



not OK in standard FEM

If $\Omega = [0,1]^2$ unit square with uniform triangulation

Piecewise linear tent functions, give $U_i = u_h(x_i)$ as in FD



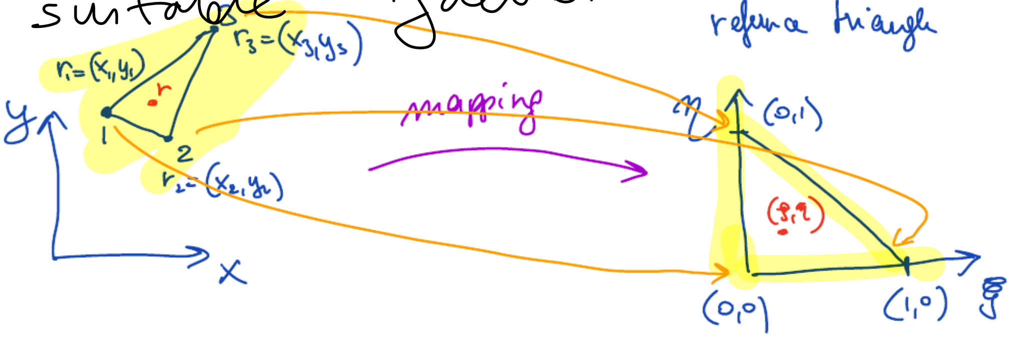
$\phi_{ij}(x,y)$

For the Laplacian,

$$A_{ij} = \int \nabla\varphi_i \cdot \nabla\varphi_j \, dx\, dy$$

is the standard FD 5pt Laplacian, and so just as ill-conditioned as for FD methods: Efficient iterative & linear solvers are based on geometric or algebraic multigrid method (AMG)

In FEM, typically things are precomputed for a <u>reference triangle</u>, and results are mapped to each triangle of the grid using suitable Jacobians.



$$r=(x,y)=\underbrace{(1-\xi-\eta)}_{\psi_1(\xi,\eta)}r_1+\underbrace{\xi}_{\psi_2(\xi,\eta)}r_2+\underbrace{\eta}_{\psi_3(\xi,\eta)}r_3$$

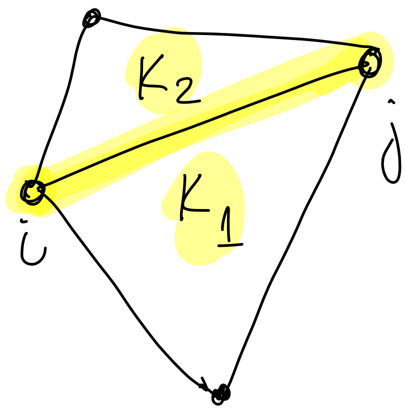
Consider trafo: $(\xi,\eta)\longmapsto r=(x,y)$

$$J=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}$$

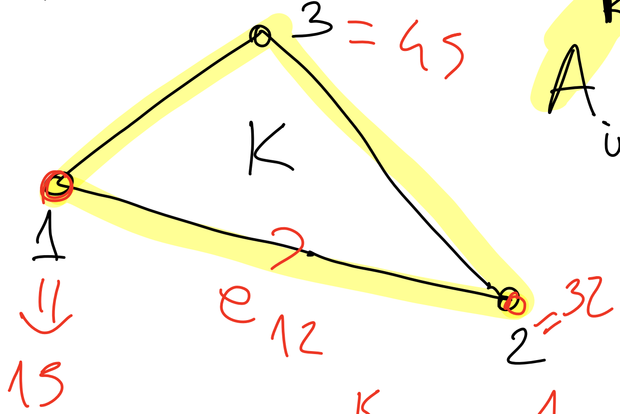
$$|J| = 2A_{123} \leftarrow \text{area of triangle}$$

Recall that for Laplacian we need $\int \nabla\psi_i\cdot\nabla\psi_j\,dx$.

But the supports of φ_i and φ_j only overlap if nodes i and j are neighbors, and therefore we get a nonzero contribution to the stiffness matrix from at most two triangles in 2D.



We therefore focus on a triangle K at a time, and assemble the stiffness matrix from triangle stiffness matrices



$$A^K_{ij} = \int_K \nabla\varphi_i \cdot \nabla\varphi_j \, dx$$

(3x3 matrix)

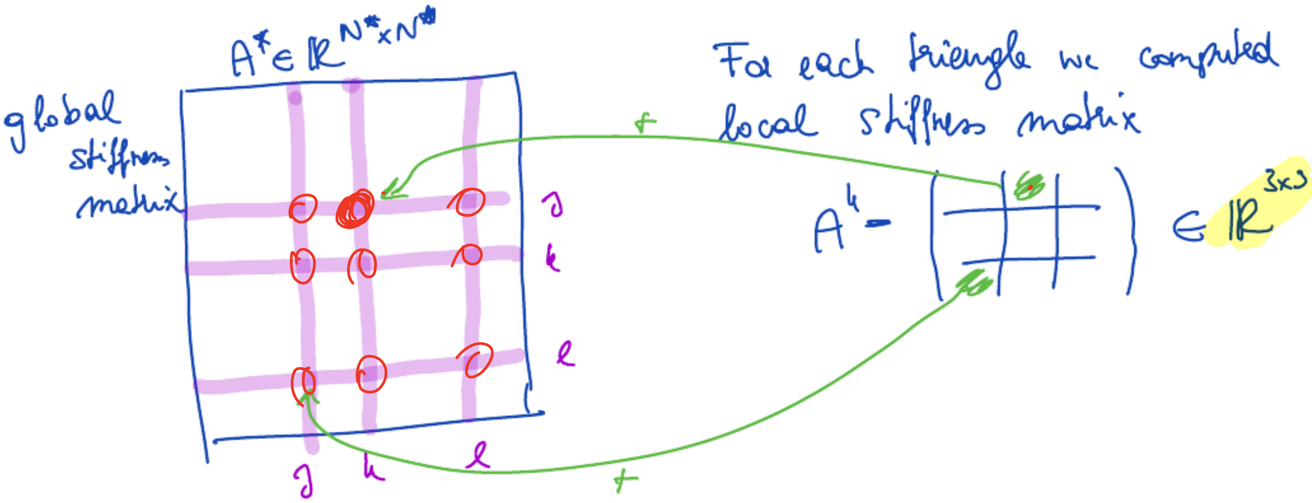
$$A^K = \frac{1}{4|A|} \begin{bmatrix} e_{23} \\ e_{13} \\ e_{23} \end{bmatrix} \begin{bmatrix} e_{23} & e_{13} & e_{23} \end{bmatrix}$$

$$A^K = \frac{1}{4A_{123}} \begin{bmatrix} |r_2-r_3|^2 & (r_2-r_3)\cdot(r_3-r_1) & (r_2-r_3)\cdot(r_1-r_2) \\ & |r_3-r_1|^2 & (r_3-r_1)\cdot(r_1-r_2) \\ \text{Symmetric} & & |r_1-r_2|^2 \end{bmatrix}$$

local stiffness matrix, corresponding to triangle K

$$A = \sum_{K=1}^{M} A^K$$

Matrix assembly summary:

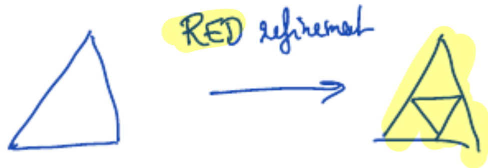


All you need is a loop over triangles and an $M \times 3$ ($3 \times M$) matrix mapping local DOFs to global DOFs

Refining grid based on a posteriori error estimate (from G. Stadler):

<u>In 2D:</u> How to refine a triangle

RED refinement

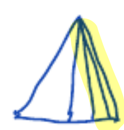


split into 4 triangles, that are shape-regular.
Problem: what to do with neighboring element?

"hanging node"

GREEN

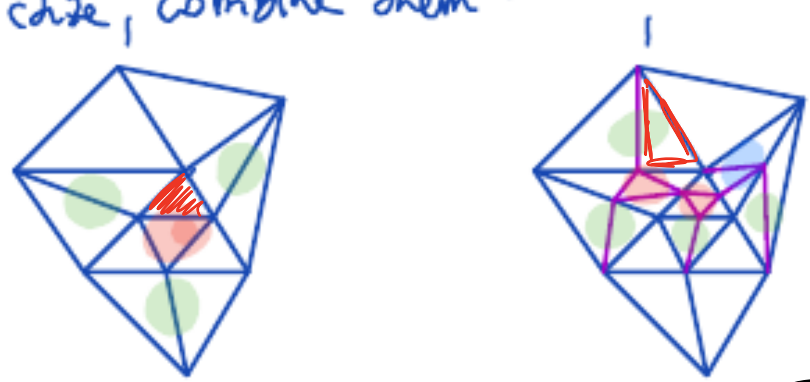
Splits into 2 triangles. Problem: iterative GREEN refinement can result in poor shape regularity:



BLUE

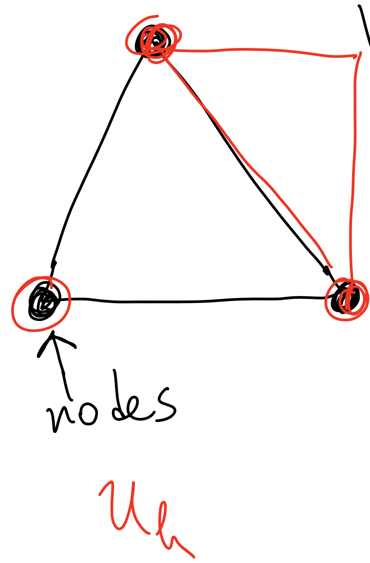
Splits one triangle into three, helps with shape regularity.

In practice, combine them:



Recall from interpolation lecture notes different elements & nodes in 2D:

① Linear triangles:



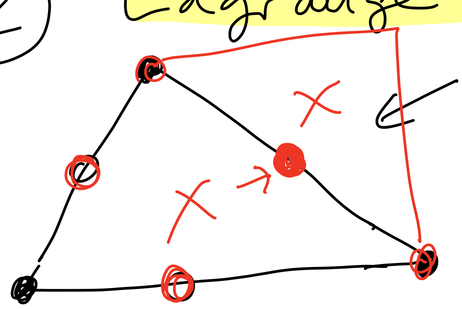
V_h = space of piecewise linear functions over triangle

Basis functions are tent functions

$$U_i = u(x_i)$$

Functions in discrete space are continuous across edges, i.e, they are continuous on Ω.

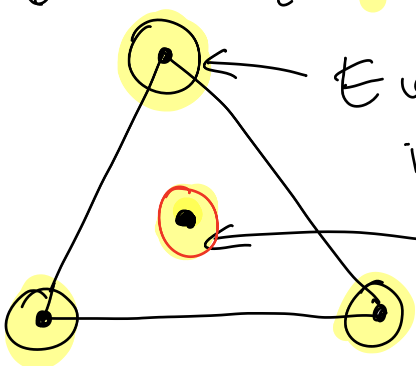
② Lagrange quadratic triangle



6 nodes, and now functions have continuous tangential derivatives along edges as well

(3) Hermite triangle

$u(x)$ is now cubic on each element (dimension $= 10$ with basis $\{1, x, y, xy, x^2, y^2, x^3, y^3, x^2y, y^2x\}$)

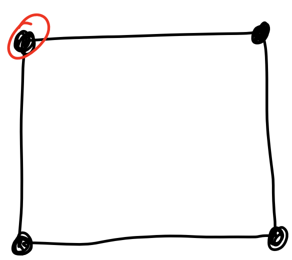


Evaluate function and its gradient $= 3$ dof's

Evaluate function only (1 dof)

$= 3 \times 3 + 1 = 10$ dofs total

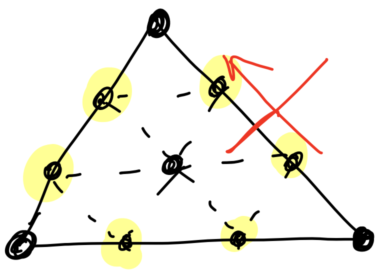
(4) Bilinear rectangle



$P = \text{span}\{1, x, y, xy\}$ (4 dofs) $\cancel{x^2}$ $\cancel{y^2}$

In general use tensor product of polynomials in x and in y; very simple but not all domains can be meshed with quadrilaterals.

⑤ Lagrange cubic triangle



Note that the global interpolant is still only C^0 since normal derivatives to an edge need not match

Sadly even Hermite triangles are <u>not</u> C^1! For higher order equations like biharmonic

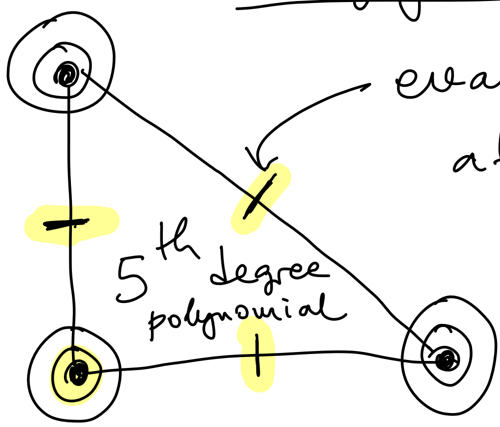
$$\nabla^4 u = f \quad , \quad \nabla u \cdot n = 0 \text{ on } \partial\Omega$$
$$u = 0 \text{ on } \partial\Omega$$

the suitable space is H_0^2.

<u>Terminology</u>: If $V_h \subset V$ (typical FEM), the FE approximation is conforming (otherwise non-conforming)

What element gives a conforming approximation to biharmonic eq.?

The Argyris triangle



evaluate normal derivative at midpoint

evaluate u, ∇u and $\nabla\nabla u$ $= 6$ dofs

(interpolation error in $H^2(\Omega)$ is $\sim h^4 \|u\|_{H^6}$)

$\#$ DOFs $= 3 \times 6 + 3 = 21$

The number of DOFs grows rapidly as one increases the order, and FEM methods can be expensive especially for vector equations

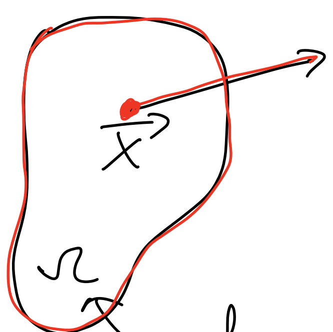
Another, probably better, alternative is to introduce a new variable

$$\begin{cases} \nabla^2 u = w \\ \nabla^2 w = f \end{cases} \rightarrow \text{mixed formulation}$$

Later we will mention
Discontinuous Galerkin (DCG)
as an alternative that avoids
increasing the number of global dofs

Linear Elasticity in 2D

Equations are similar in structure
to fluids but variable is
displacement field $u(x) \in \mathbb{R}^2$
(not velocity)



$\vec{x} + \vec{u}(x)$ new position
of point $\vec{x}$

reference configuration

Strain tensor (similar to strain rate for fluids)

$$\overleftrightarrow{e}(\vec{u}) = \frac{1}{2}\left(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^{T}\right) \in \mathbb{R}^{2\times 2}$$

Linear elasticity (small deformation)

Stress tensor

$$\overleftrightarrow{\sigma} = \overset{\langle\langle\longleftrightarrow\rangle\rangle}{L}\,\overleftrightarrow{e}$$

$$\sigma_{ij} = \sum_{k,l} L_{ijkl}\, e_{kl}$$

implied summation

Isotropic material must have

$$L_{ijkh} = \lambda\, \delta_{ij}\, \delta_{kh} + \mu\left(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\right)$$

$$\begin{cases} \mu > 0 \\ \lambda + 2\mu > 0 \end{cases}$$

Lamé parameters (like viscosity, property of solid)

$$\Rightarrow \overleftrightarrow{\sigma} = 2\mu\, \overleftrightarrow{e} + \lambda\, \mathrm{Tr}(\overleftrightarrow{e})\, \overleftrightarrow{I}$$

trace

Strong form of PDE

$$\vec{\nabla} \cdot \overleftrightarrow{\sigma} = \vec{f} \Leftarrow \text{body force (applied force)}$$

BCs are just like for Navier-Stokes: Specify one BC for <u>normal</u> direction (either $\vec{u}$ or $\overleftrightarrow{\sigma} \cdot \vec{n}$) and one for tangential (either $\vec{u}$ or $\overleftrightarrow{\sigma} \cdot \vec{t}$)

For essential BC (Dirichlet)

$\vec{u}(\partial\Omega) = \vec{0}$ we have an

<u>Energy formulation</u>:

$$\min_{u \in (H_0^1(\Omega))^2} \int_\Omega \Big(\overbrace{\tfrac{1}{2} \sigma : e}^{\text{elastic energy}} - \underbrace{f \cdot u}_{\text{work done}} \Big) dx$$

$$\sigma : e = \sum_{i,j} \sigma_{ij} e_{ij} \quad \text{(double contraction)}$$

Weak formulation

$$\begin{cases} \nabla \cdot \sigma = f & \text{in } \Omega \\ \boxed{\sigma \cdot n = g \quad \text{on } \partial\Omega_2} \\ u = 0 & \text{on } \partial\Omega_1 \end{cases}$$

natural

essential BC

fix

Find $u \in \left(\bar{H}^1_0\right)^2$ s.t.

$$a(u, v) = \ell(v) \quad \forall v \in \left(\bar{H}^1_0\right)^2$$

$$u(\partial\Omega_1) = v(\partial\Omega_1) = 0$$

Where as before

$$a(u, v) = \frac{1}{2} \int_\Omega e(v) : L\, e(u)\, dx$$

weakly imposed BC

$$\ell(v) = \int_\Omega f \cdot v\, dx + \boxed{\int_{\partial\Omega_2} g \cdot v\, ds}$$

Using a triangulation with linear hat basis functions

$$\{\eta_1, \dots, \eta_{2N}\} = \{ \varphi_1 e_1, \varphi_1 e_2, \varphi_2 e_1, \varphi_2 e_2, \dots \}$$

$$\Rightarrow \quad A U = F$$

$$U = \{ U_1^x, U_1^y, \dots, U_N^x, U_N^y \}$$

$$\equiv \{ U_1, U_2, \dots, U_{2N} \}$$

$$A_{k\ell} = \int_\Omega e(\eta_k) : L e(\eta_\ell)\, dx$$

$$F_k = \int_\Omega f \eta_k \, dx + \int_{\partial\Omega_2} g \cdot \eta_k \, ds$$

Explicit formulas can be obtained for linear triangles for the $(3 \cdot 2)^2 = 6 \times 6$ local / triangle stiffness matrix

Introduce

$$\gamma(u) = \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ 2e_{12}(u) \end{pmatrix}$$

$$\Rightarrow \quad e(v):Le(u) = \gamma^T C \gamma \quad \text{where}$$

$$C = \begin{pmatrix} \lambda+2\mu > 0 & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu > 0 \end{pmatrix}$$

On a triangle

$$\gamma(u_h) = \underbrace{\begin{bmatrix} \varphi_{1x} & 0 & \varphi_{2x} & 0 & \varphi_{3x} & 0 \\ 0 & \varphi_{1y} & 0 & \varphi_{2y} & 0 & \varphi_{3y} \\ \varphi_{1y} & \varphi_{1x} & \varphi_{2y} & \varphi_{2x} & \varphi_{3y} & \varphi_{3x} \end{bmatrix}}_{R} \begin{bmatrix} u_1 \\ \vdots \\ u_6 \end{bmatrix}$$

Local DOFs on triangle

$$A^K = |K| \, R^T C R \quad (\text{clearly SPD})$$

A more tricky example is

Stokes flow

$$\begin{cases} \nabla p = -\nabla\cdot\sigma + f \\ \nabla\cdot u = 0 \leftarrow \text{hard} \end{cases} \quad \text{in } \Omega$$

$$\sigma = -\eta\left(\nabla u + \nabla u^T\right), \quad \eta = \text{const}$$

Now $\vec{u}$ is velocity

What we say here also applies to Navier-Stokes eqs.

Energy formulation:

$$\begin{cases} \min\limits_{v \in V} \dfrac{\eta}{2}\displaystyle\int_\Omega \overset{(\nabla u + \nabla u^T)}{\nabla v : \nabla v}\, dx + \int_\Omega f\cdot v\, dx \\ \text{s.t. } \boxed{\nabla\cdot v = 0} \leftarrow p \text{ is Lagrange multiplier for constraint} \end{cases}$$

Weak form involves different spaces for v and p: mixed FE.

Find $u \in (H_0^1(\Omega))^d$, $p \in L^2(\Omega)$
up to a constant

s.t.

$(\nabla \cdot v, q)_{L_2}$

$$\begin{cases} a(u,v) + b(v,p) = F(v) \\ b(u,q) = 0 \end{cases}$$

H^1 $\quad$ L_2

mixed bilinear form

$\forall (v, q) \in V \times Q$

saddle-point system

$$\begin{pmatrix} A+\alpha M & B^T \\ B & \emptyset \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

NS

Where $BU = 0$ defines the kernel space K (discretely divergence-free velocity fields)

$\# \nabla \cdot u = 0$

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$$a(u, v) = \eta \int_\Omega (\nabla u + \nabla^T u) : (\nabla v + \nabla v^T) \, dx$$

$$b(v, p) = \int_\Omega p \, (\nabla \cdot v) \, dx$$

Saddle-point system

$$v \cdot \left[\, A u + B^T p = f \right.$$

$$\Rightarrow \quad v^T A u + \underbrace{\cancel{(B v)}}_{\text{zero}}{}^T p = v^T f$$

$$\Rightarrow \quad \begin{cases} v^T A u = v^T f \\ \forall v \in K \\ \text{(variational problem on } K\text{)} \end{cases}$$

A key feature of Stokes flow is that the saddle-point system must be solvable & well-conditioned as $h \to 0$

Mathematically, this is expressed as the inf-sup condition (also called LBB = Ladyszenskaya, Brezzi, Babushka):

$$\inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \gamma > 0$$

This condition means that pressure space Q_h cannot be "too large", since otherwise there will be some $q_h \in Q_h$ (a spurious or "parasitic mode") that will make the sup be zero.

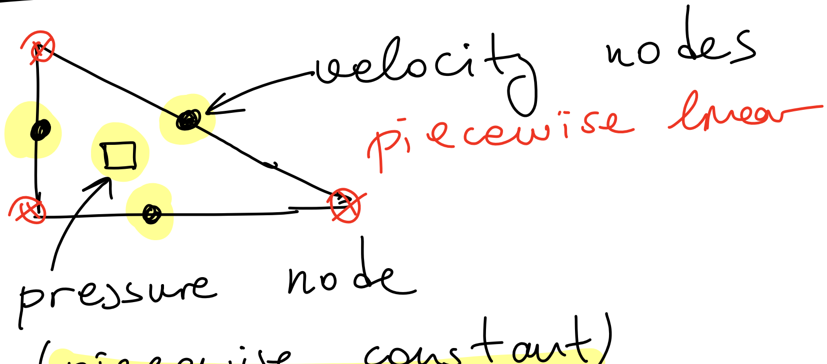
V_h and Q_h must be chosen together not independently

Using linear triangles for both pressure & velocity is NOT inf-sup stable and does not converge for Stokes.

Heuristically, the polynomial degree for pressure should be one lower than velocity.

Examples of stable elements:

Crouzeix–Raviart element



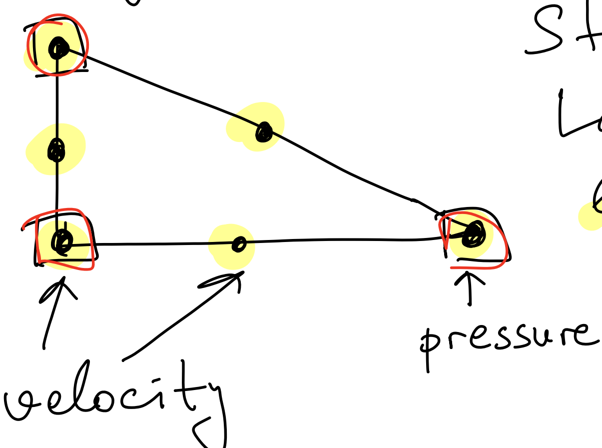
This is a non-conforming element since velocity is not continuous accross edges so $V_h \not\subset H^1$

It is in some sense the equivalent of the MAC or staggered grid for triangles (but first order for velocity)

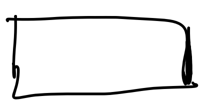
A more standard stable element is the

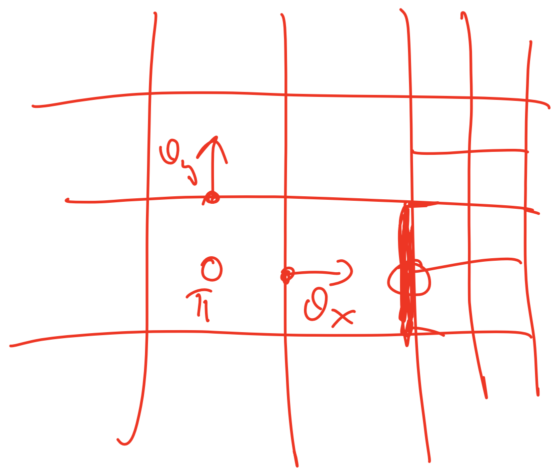
Taylor-Hood element

Standard quadratic Lagrange for velocity, linear for pressure



Already quite a bit more expensive than MAC!

Also stable is V_h order $k \geq 2$ polynomial for  cells, pressure degree $k-2$ discontinuous



ϑ_y
ϑ
ϑ_x