

Limiters

A. DONEY, Spring 2021

All of the second-order schemes for advection we studied (Lax-Wendroff, Fromm, Beam-Warming) had **artificial dispersion** and therefore introduced **oscillations, undershoots or overshoots**, especially near discontinuities such as shocks. A lot of effort has gone over the years in developing so-called **High-Resolution** or **Shock-Capturing** schemes for hyperbolic equations,

①

as a way to prevent the
undesired oscillations, and preserve
discontinuities such as shocks.

We will not be able to
cover this broad topic, and will
instead focus on one component
that is key to all such
schemes: Limiters

Loosely stated, the goal of
limiters is to prevent the formation
Spurious extrema (minima/maxima)
in the solution when the
true solution is expected to
be monotonic. Often called
"monotonicity preserving"

(we will not study the theory behind this in this class)

A fundamental result in the field is the **Godunov order barrier theorem**:

Linear schemes for advection do not generate new minima/maxima can be at most **first-order accurate**.

Second-order monotonicity-preserving schemes MUST be NONLINEAR — Even if the PDE is linear!

Note: one can similarly show that schemes for diffusion that are free of spurious oscillations are at most **second-order**!

(3)

Once again we see that upwinding + centered 3rd diffusion is the most robust linear scheme.

We need a way to introduce non-linearity into our schemes.

There are two equivalent ways to present the idea:

- slope limiters (more geometric & intuitive but harder to implement)
- flux limiters (more algebraic but easiest to implement)

Let's start with

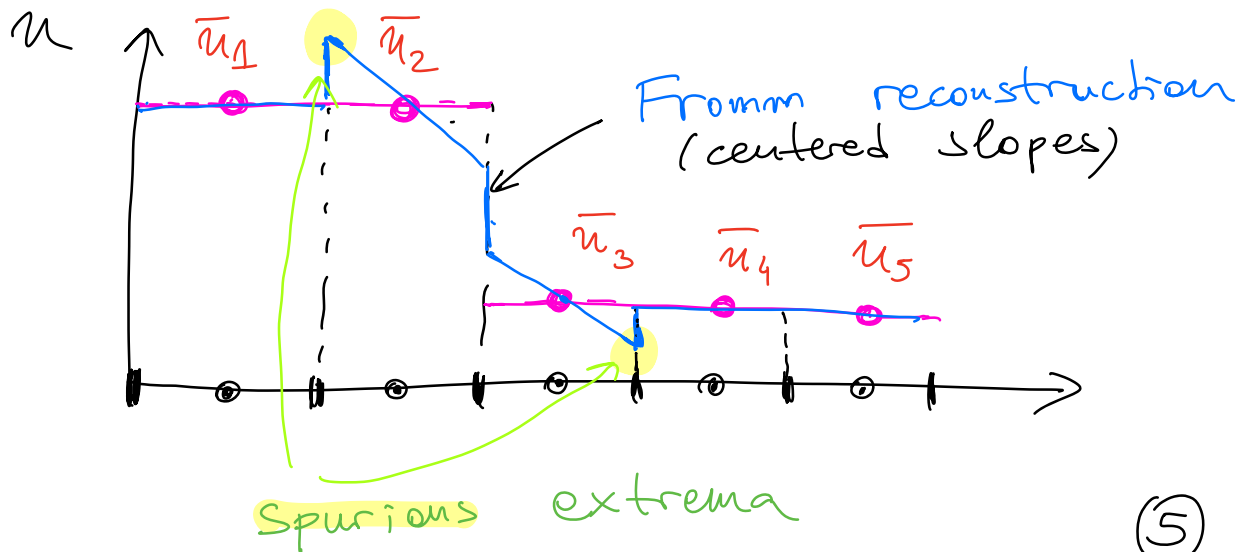
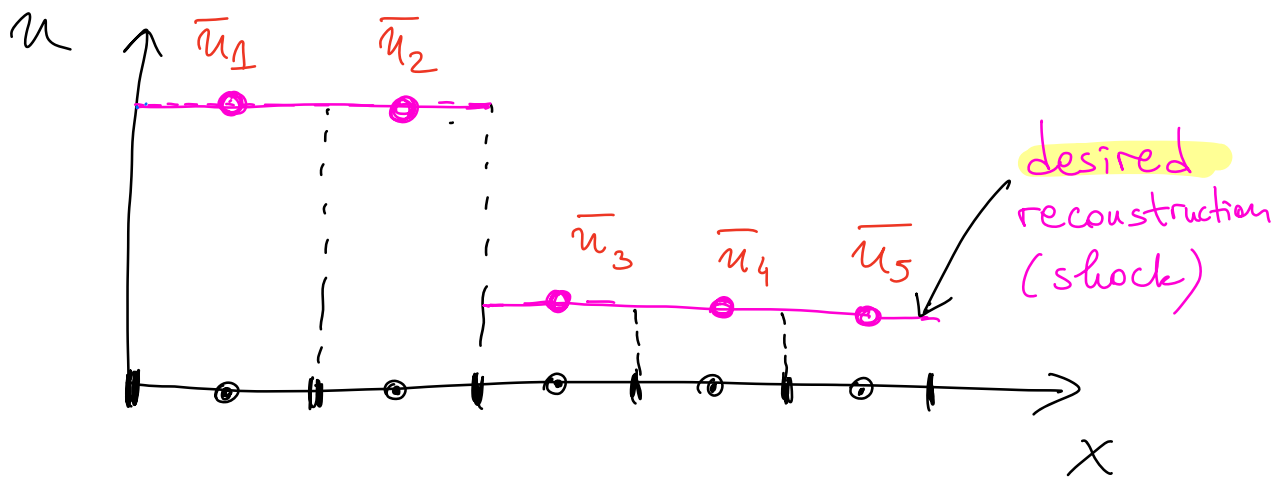
SLOPE LIMITERS

Recall that one approach was:

Reconstruct \rightarrow Advect \rightarrow Average
to cells (4)

This suggests that what we want is that:

The reconstruction should not have **spurious** local minima and maxima



Instead of setting the slope to $S_j = \frac{u_{j+1} - u_{j-1}}{2h}$ in

cells 2 & 3 we should have set it to **zero!**

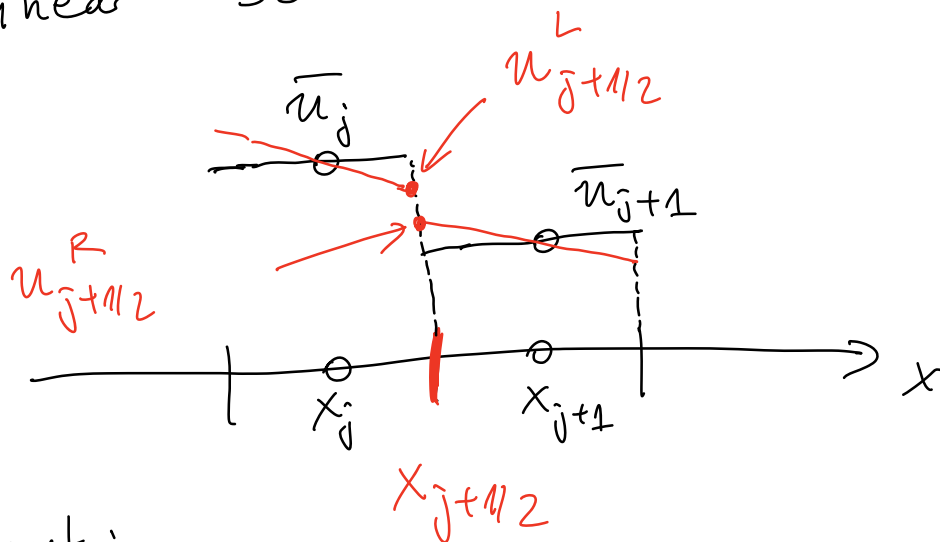
Setting the slope to zero is a way to limit the magnitude of the slopes near discontinuities.

Zero slope means going back to simple upwinding, i.e., back to first order. But this is OK since near a discontinuity the solution is not smooth anyway & at best we can expect 1st order accuracy anyway

(6)

Idea: In regions/cells where the solution is locally smooth, use 2nd order scheme. Where it isn't, go back to simple upwinding.

This is non linear since we use the solution itself to change the coefficients in the linear scheme.



Want:

$$\min(\bar{u}_j, \bar{u}_{j+1}) \leq u_{j+1/2}^{(L/R)} \leq \max(\bar{u}_j, \bar{u}_{j+1}) \quad (7)$$

unless \bar{u}_j is a local extremum
 (e.g., $\bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j-1}$)

One way to accomplish this
 is to set:

$$S_i = \text{minmod} \left(\underbrace{\frac{\bar{u}_i - \bar{u}_{i-1}}{h}}_{\text{Bear-Warming (if } a > 0)}, \underbrace{\frac{\bar{u}_{i+1} - \bar{u}_i}{h}}_{\text{Lax-Wendroff}} \right)$$

where $\text{minmod}(a, b) = \begin{cases} 0 & \text{if } ab < 0 \\ \text{otherwise:} \\ a & \text{if } |a| < |b| \\ b & \text{if } |b| < |a| \end{cases}$

So this selects the slope of
 smaller magnitude between the
 upwinded and downwinded slopes,

⑧

otherwise it sets slope = zero
if \bar{u}_j is a local extremum.

This is called the **minmod**
slope limiter.

A better one in practice is
the so-called **MC limiter**
of van Leer:

$$S_i = \text{minmod} \left[\frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2h}, 2 \left(\frac{\bar{u}_i - \bar{u}_{i-1}}{h} \right), 2 \left(\frac{\bar{u}_{i+1} - \bar{u}_i}{h} \right) \right]$$

Fromm method,
2nd order

in consistent
slope estimates
 \Rightarrow first order

If the solution is smooth, then we expect

$$\frac{u_{i+1} - u_i}{2h} \approx \frac{u_{i+1} - u_i}{h} \approx \frac{u_i - u_{i-1}}{h}$$

and so the MC limiter will choose the centered (Fromm) slope and be 2nd order.

Only if the slope changes a lot or changes sign near cell i then we switch to first-order schemes.

If we fix the reconstruction slopes with the MC limiter, then advection & average, we will not get spurious extrema

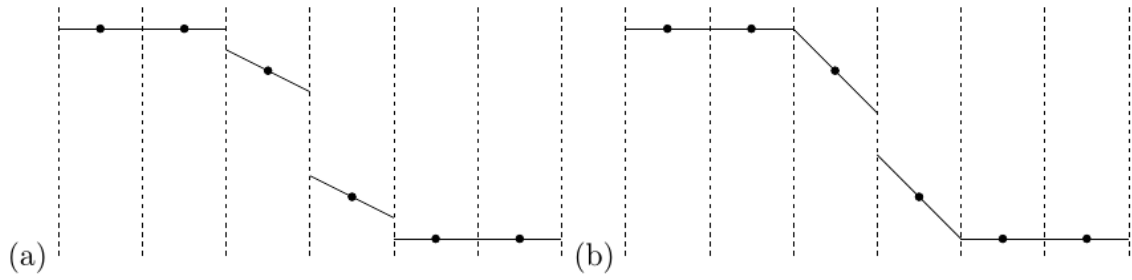


Fig. 6.5. Grid values Q^n and reconstructed $\tilde{q}^n(\cdot, t_n)$ using (a) minmod slopes, (b) superbee or MC slopes. Note that these steeper slopes can be used and still have the TVD property.

(LeVeque FV book)

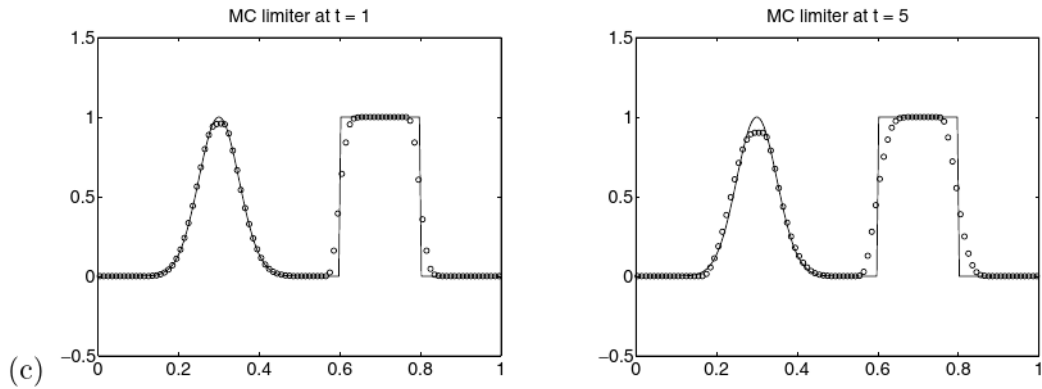


Fig. 6.2. Tests on the advection equation with different high-resolution methods, as in Figure 6.1:

11

Flux limiters

In practice, it is often easier & more flexible to recast slope limiters as flux limiters (same effect, just a reformulation).

The idea here is to switch focus from cells to faces, and do limiters on the faces/fluxes.

We begin by rewriting our 2nd order linear reconstruction in terms of slopes at faces of the grid, $\bar{S}_{i-1/2}$ instead of S_i as before.

(12)

Define the jump:

$$\Delta u_{i-1/2} = \bar{u}_i - \bar{u}_{i-1}$$

so that a slope estimate on face $i-1/2$ is $\Delta u_{i-1/2}/h$.

It is easy to see that Lax-Wendroff can be rewritten with fluxes:

$$F_{i-1/2}^{n+1/2} = \underbrace{\begin{bmatrix} a_{i-1/2}^{n+1/2} \\ i-1/2 \end{bmatrix}}_{\text{upwind flux}} \left(\bar{u}_i^n + \frac{\Delta t}{2} g_i^n \right) \overset{\text{source}}{\sim}$$

$$+ \begin{bmatrix} a_{i-1/2}^{n+1/2} \\ i-1/2 \end{bmatrix}^+ \left(\bar{u}_{i-1}^n + \frac{\Delta t}{2} g_{i-1}^n \right)$$

(13)

second-order correction

$$+ \frac{1}{2} \left| a_{\bar{i}-1/2}^{n+1/2} \right| (1 - \gamma_{\bar{i}+1/2}^{n+1/2}) \delta_{\bar{i}-1/2}^n$$

CFL number $\gamma_{\bar{i}+1/2}^{n+1/2} = \left| \frac{a_{\bar{i}-1/2}^{n+1/2} \cdot \tau}{h} \right|$

Slope estimate on face
 $= \delta_{\bar{i}-1/2}^n / h$

and recall that the source:

$$g_j = - \underbrace{u_j \left(\frac{a_{\bar{j}+1/2}^{n+1/2} - a_{\bar{j}-1/2}^{n+1/2}}{h} \right)}_{\text{From } -u a_x} + (\text{source})_j$$

e.g. diffusion

From $-u a_x$

The idea is to now set

$$\delta_{\bar{i}-1/2}^n = \text{a limited form of } \Delta u_{\bar{i}-1/2}^n$$

$$\Delta u_{\bar{i}-1/2} = \bar{u}_{\bar{i}} - \bar{u}_{\bar{i}-1}$$

If we set

$$\delta_{\bar{t}-1/2}^n = \Delta u_{\bar{t}-1/2}^n$$

we get Lax-Wendroff.

Instead, let's set

$$\delta_{\bar{t}-1/2}^n = \varphi(\theta_{\bar{t}-1/2}) \Delta u_{\bar{t}-1/2}^n$$

$$\theta_{\bar{t}-1/2} = \frac{\Delta u_{I-1/2}}{\Delta u_{i-1/2}}$$

upwind face
from i

$$I = \begin{cases} \bar{t}-1 & \text{if } a_{\bar{t}-1/2} > 0 \\ \bar{t}+1 & \text{if } a_{\bar{t}-1/2} < 0 \end{cases}$$

Smoothness indicator:

$\theta \approx 1$ means solution is locally smooth on grid (15)

Flux limiter function $\psi(\theta)$:

Linear methods: (LeVeque FVM book)

upwind : $\phi(\theta) = 0,$

Lax-Wendroff : $\phi(\theta) = 1,$

Beam-Warming : $\phi(\theta) = \theta,$

Fromm : $\phi(\theta) = \frac{1}{2}(1 + \theta).$ ✓

(6.39a)

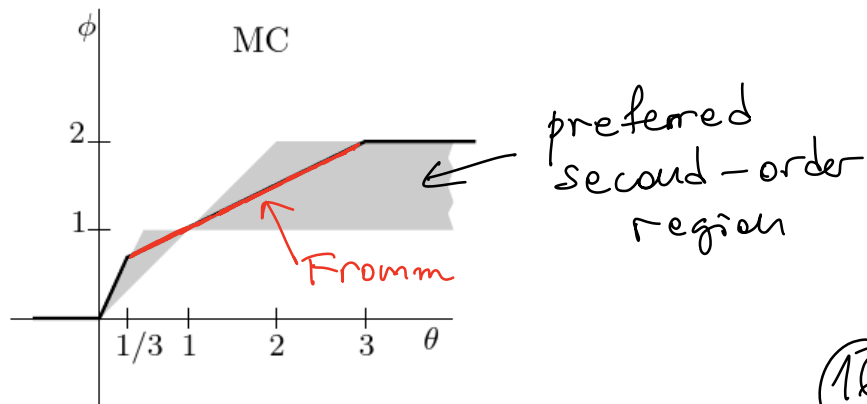
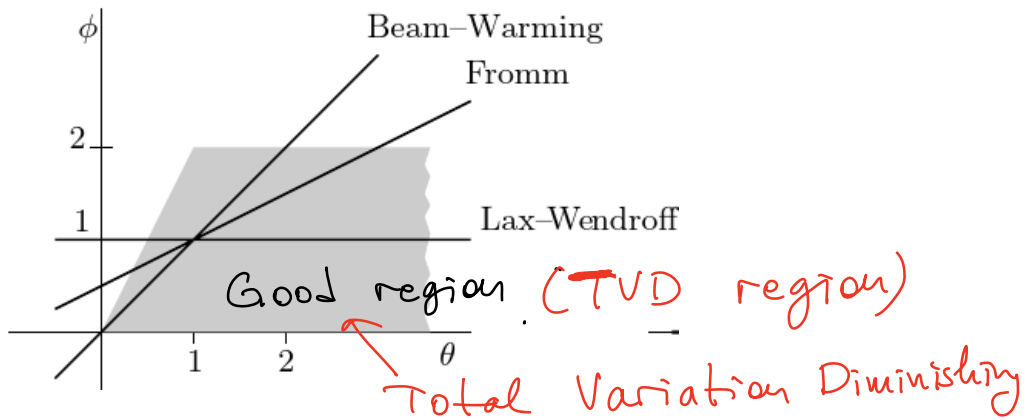
High-resolution limiters:

minmod : $\phi(\theta) = \text{minmod}(1, \theta),$

superbee : $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)),$

✓ MC : $\phi(\theta) = \max(0, \min((1 + \theta)/2, 2, 2\theta))$ (6.39b)

van Leer : $\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}.$



Code demo in class

Bad news: All strictly limited (technically, total variation diminishing) methods degenerate to first-order near maxima: They clip or flatten the maximum.

There are solutions to this, for example, PPM quadratic reconstruction limiter:

see final project by Wenjun Zhang linked on webpage (also a great example of a final project for class)

Limiters can also be used with MOL methods.

We always write flux as upwind + 2nd order correction:

$$F_{j+1/2}(\bar{u}, t) = a_{j+1/2}^{\rightarrow 0}(t) \cdot u_{j+1/2}(\bar{u})$$

$$u_{j+1/2} = \bar{u}_j + \Psi(\theta_j) (\bar{u}_{j+1} - \bar{u}_j)$$

where smoothness indicator

$$\theta_j = \frac{\bar{u}_j - \bar{u}_{j-1}}{\bar{u}_{j+1} - \bar{u}_j}$$

Observe:

$$\Psi(\theta) = \frac{1}{3} + \frac{\theta}{6} \text{ gives}$$

3rd order upwind biased!

(18)

$$\begin{aligned}
u_{j+1/2} &= \bar{u}_j + \frac{1}{3} (\bar{u}_{j+1} - \bar{u}_j) \\
&\quad + \frac{1}{6} (\bar{u}_j - \bar{u}_{j-1}) = \\
&= \frac{5}{6} \bar{u}_j + \frac{1}{3} \bar{u}_{j+1} - \frac{1}{6} \bar{u}_{j-1}
\end{aligned}$$

which is the same as what we wrote before

Use Koren limiter

$$\psi = \max(0, \min(1, \frac{1}{3} + \frac{\theta}{6}, \theta))$$

which is the equivalent of MC limiter for 3rd order upwind biased.

Must be combined with special "strong stability preserving" (SSP) RK3 integrator (see IMEX lecture) (19)