Error (Empirical) Estimates

~~"refined grid"~~
~~"reference solution"~~

$$\begin{cases} u_h(x) = u_{\text{true}}(x) + h^2 e(x) + O(h^3) \\ u_{2h}(x) = u_{\text{true}} + (2h)^2 e(x) + O(h^3) \end{cases}$$

$O(h^2)$

$$u_h - h^2 e \approx u_{\text{true}} + O(h^3)$$

Limiters

A. Donev, Spring 2021

All of the second-order schemes for advection we studied (Lax-Wendroff, Fromm, Beam-Warming) had artificial dispersion and therefore introduced oscillations, undershoots or overshoots, especially near discontinuities such as shocks. A lot of effort has gone over the years in developing so-called

High-Resolution or

Shock-Capturing

schemes for hyperbolic equations,

as a way to prevent the undesired oscillations. and <u>preserve</u> discontinuities such as shocks.

We will not be able to cover this broad topic, and will instead focus on one component that is key to <u>all</u> such schemes: <u>Limiters</u>

Loosely stated, the goal of limiters is to prevent the formation spurious extrema (minima/maxima) in the solution when the true solution is expected to be monotonic. Often called "monotonicity preserving"

(we will not study the theory behind this in this class)

A fundamental result in the field is the Godunov order barrier theorem:

Linear schemes for advection that do not generate new minima/maxima can be at most first-order accurate.

Second-order monotinicity-preserving schemes MUST be NONLINEAR — Even if the PDE is linear!

Note: one can similarly show that schemes for diffusion that are free of spurious oscillations are at most second-order!

Once again we see that upwinding + centered 3^{pt} diffusion is the <u>most robust</u> linear scheme.

We need a way to introduce non-linearity into our schemes.

There are two <u>equivalent</u> ways to present the idea:

- slope limiters (more geometric & intuitive but harder to implement)
- flux limiters (more algebraic but easiest to implement)

Let's start with

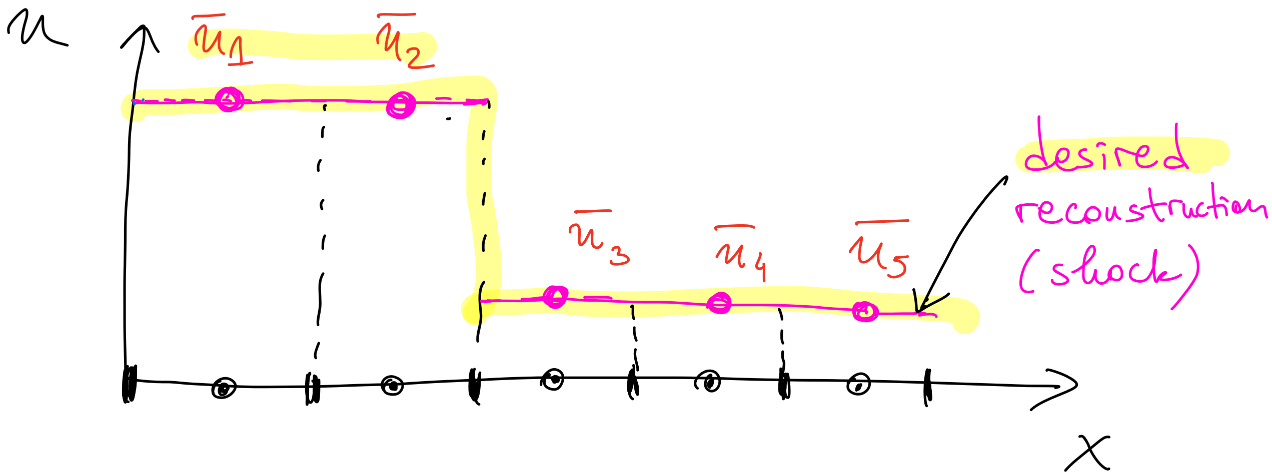
SLOPE LIMITERS

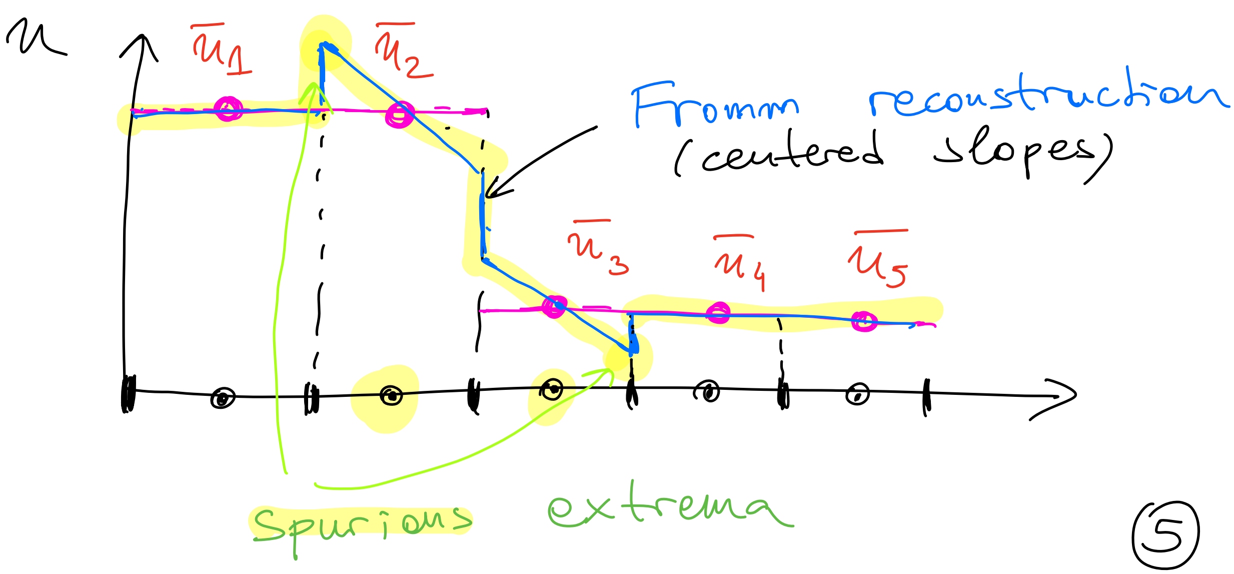
Recall that one approach was:

Reconstruct → Advect → Average to cells

This suggests that what we want is that:

$\boxed{\text{The reconstruction should not have spurious local minima and maxima}}$



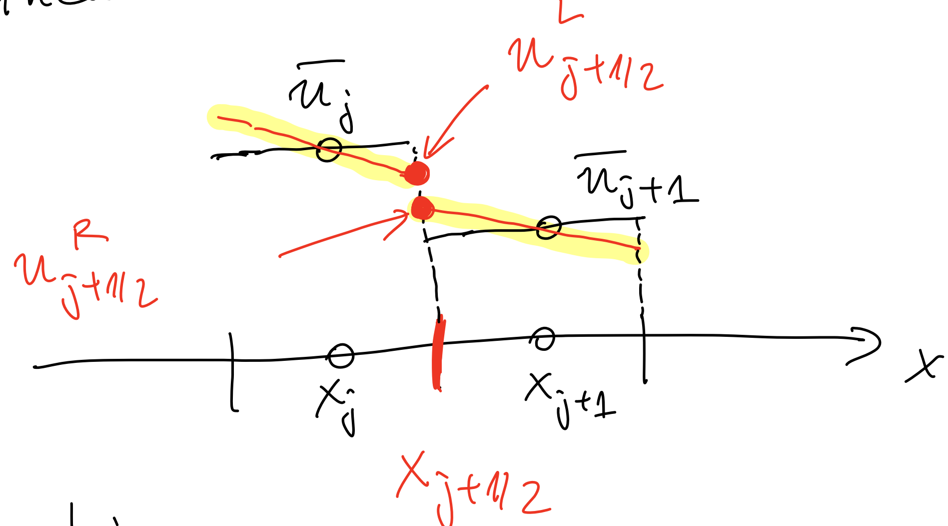


Instead of setting the slope to $S_j = \frac{u_{j+1} - u_{j-1}}{2h}$ in cells 2 & 3 we should have set it to zero!

Setting the slope to zero is a way to limit the magnitude of the slopes near discontinuities. Zero slope means going back to simple upwinding, i.e., back to first order. But this is OK since near a discontinuity the solution is not smooth anyway & at best we can expect 1st order accuracy anyway

Idea: In regions/cells where the solution is locally smooth, use 2nd order scheme. Where it isn't, go back to simple upwinding.

This is nonlinear since we use the solution itself to change the coefficients in the linear scheme.



Want:

$$\min(\bar{u}_j, \bar{u}_{j+1}) \le u^{(L/R)}_{j+1/2} \le \max(\bar{u}_j, \bar{u}_{j+1}) \quad (7)$$

Fix

unless $\bar{u}_j$ is a local extremum

(e.g., $\bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j-1}$)

One way to accomplish this is to set:

$$S_i = \text{minmod}\Big(\underbrace{\frac{\bar{u}_i - \bar{u}_{i-1}}{h}}_{\substack{\text{Beam-Warming} \\ (\text{if } a>0)}}, \underbrace{\frac{\bar{u}_{i+1} - \bar{u}_i}{h}}_{\substack{\text{Lax-} \\ \text{Wendroff}}}\Big)$$

where

$$\text{minmod}(a,b) = \begin{cases} 0 & \text{if } ab < 0 \\ \text{otherwise:} & \\ a & \text{if } |a| < |b| \\ b & \text{if } |b| < |a| \end{cases}$$

So this selects the slope of smaller magnitude between the upwinded and downwinded slopes.

otherwise it sets slope = zero if $\bar{u}_j$ is a local extremum.

This is called the minmod slope limiter.

A better one in practice is the so-called MC limiter of van Leer:

$$S_i = \text{minmod}\left[\frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2h},\ 2\left(\frac{\bar{u}_i - \bar{u}_{i-1}}{h}\right),\ 2\left(\frac{\bar{u}_{i+1} - \bar{u}_i}{h}\right) \right]$$

Fromm method, 2nd order where u is smooth

in consistent slope estimates $\Rightarrow$ first order

If the solution is smooth, then we expect

$$\frac{u_{i+1}-u_i}{2h} \simeq \frac{u_{i+1}-u_i}{h} \simeq \frac{u_i-u_{i-1}}{h}$$

and so the MC limiter will choose the centered (Fromm) slope and be 2nd order. Only if the slope changes a lot or changes sign near cell i then we switch to first-order schemes.

If we fix the reconstruction slopes with the MC limiter, then advect & average, we will <u>not</u> get spurious extrema

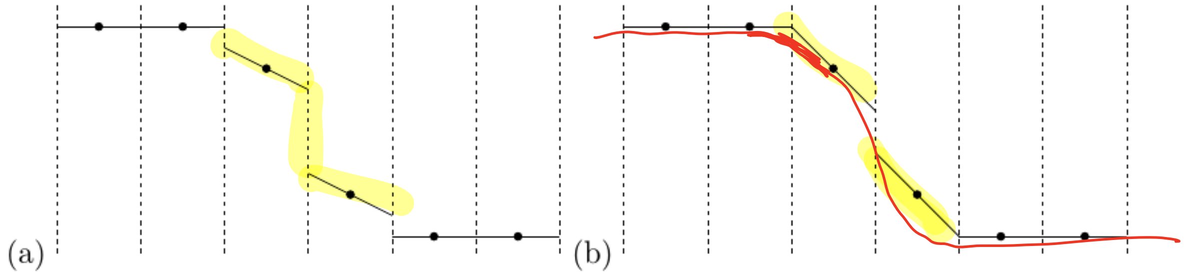


Fig. 6.5. Grid values Q^n and reconstructed $\tilde{q}^n(\cdot, t_n)$ using (a) minmod slopes, (b) superbee or MC slopes. Note that these steeper slopes can be used and still have the TVD property.

(LeVeque FV book)

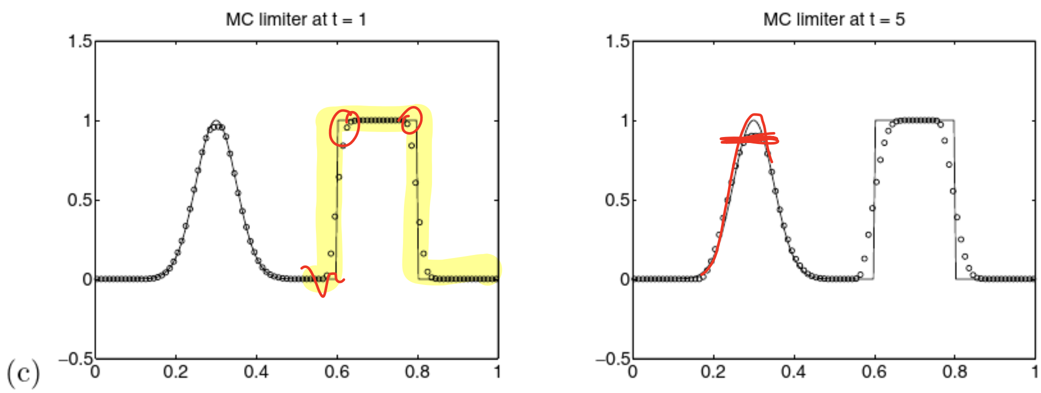


Fig. 6.2. Tests on the advection equation with different high-resolution methods, as in Figure 6.1:

Flux limiters

In practice, it is often easier & more flexible to recast slope limiters as flux limiters (same effect, just a reformulation).

The idea here is to switch focus from cells to faces, and do limiters on the faces/fluxes.

We begin by rewriting our 2^{nd} order linear reconstruction in terms of slopes at faces of the grid, $\delta_{i-1/2}$ instead of S_i as before.

Define the jump:

$$\Delta u_{i-1/2} = \bar{u}_i - \bar{u}_{i-1}$$

so that a slope estimate on face $i-1/2$ is $\Delta u_{i-1/2}/h$.

It is easy to see that Lax-Wendroff can be rewritten with fluxes:

$$F^{n+1/2}_{i-1/2} = \underbrace{\left[a^{n+1/2}_{i-1/2}\right]^{-}\left(\bar{u}^n_i + \overbrace{\frac{\Delta t}{2} g^n_i}^{\text{source}}\right)}_{\text{upwind flux}}$$

Fix

$$+ \left[a^{n+1/2}_{i-1/2}\right]^{+}\left(\bar{u}^n_{i-1} + \frac{\Delta t}{2} g^n_{i-1}\right)$$

second-order correction

$$+\frac{1}{2}\left|a^{n+1/2}_{i-1/2}\right|\left(1-\nu^{n+1/2}_{i+1/2}\right)\underbrace{\delta^{n}_{i-1/2}}_{\substack{\text{slope}\\ \text{estimate}\\ \text{on face}}} \neq u_i - u_{i-1}$$

CFL number

$$\nu^{n+1/2}_{i+1/2} = \left|\frac{a^{n+1/2}_{i-1/2}\cdot\bar{\tau}}{h}\right|$$

and recall that the source:

$$g_j = \underbrace{-\bar{u}_j\left(\frac{a^{n+1/2}_{j+1/2}-a^{n+1/2}_{j-1/2}}{h}\right)}_{\text{From } -u\,a_x} + (\text{source})_j$$

e.g. diffusion

The idea is to now set

$\delta^{n}_{i-1/2}$ = a limited form of $\Delta u^{n}_{i-1/2}$

$$\Delta u_{i-1/2} = \bar{u}_i - \bar{u}_{i-1}$$

If we set

$$\delta^n_{i-1/2} = \Delta u^n_{i-1/2}$$

we get Lax-Wendroff.

Instead, let's set

$$\delta_{i-1/2} = \boxed{\varphi(\theta_{i-1/2})} \underline{\underline{\Delta u_{i-1/2}}}$$

$$\theta_{i-1/2} = \frac{\Delta u_{I-1/2}}{\underline{\underline{\Delta u_{i-1/2}}}}$$

upwind face from i

$$I = \begin{cases} i-1 & \text{if } a_{i-1/2} > 0 \\ i+1 & \text{if } a_{i-1/2} < 0 \end{cases}$$

smoothness indicator:

$\theta \simeq 1$ means solution is locally smooth on grid

Flux limiter function $\varphi(\theta)$:

Linear methods: (Le Veque FVM book)

$$
\begin{aligned}
\text{upwind}: &\quad \phi(\theta) = 0, \\
\text{Lax–Wendroff}: &\quad \phi(\theta) = 1, \\
\text{Beam–Warming}: &\quad \phi(\theta) = \theta, \\
\text{Fromm}: &\quad \boxed{\phi(\theta) = \frac{1}{2}(1+\theta).}
\end{aligned}
\tag{6.39a}
$$

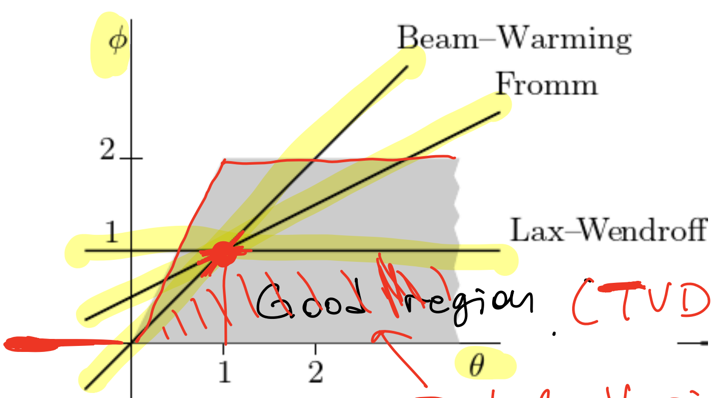
✓

High-resolution limiters:

$$
\begin{aligned}
\text{minmod}: &\quad \phi(\theta) = \text{minmod}(1, \theta), \\
\text{superbee}: &\quad \phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)), \\
&\boxed{\text{MC}: \quad \phi(\theta) = \max(0, \min((1+\theta)/2, 2, 2\theta))} \\
\text{van Leer}: &\quad \phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}.
\end{aligned}
\tag{6.39b}
$$

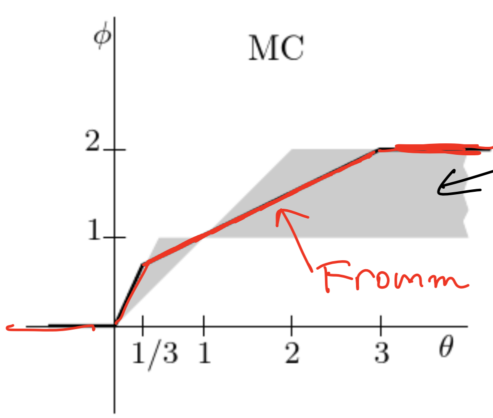
✓

smooth



(TVD region)

Total Variation Diminishing



prefered second-order region

Code demo in class

Bad news: All strictly limited (technically, total variation diminishing) methods degenerate to first-order near maxima: They clip or flatten the maximum.

There are solutions to this, for example, PPM quadratic reconstruction limiter: see final project by Wenjun Zhang linked on webpage (also a great example of a final project for class)

Limiters can also be used with MOL methods.
We always write flux as upwind + 2nd order correction:

$$F_{j+1/2}(\bar{u}, t) = \overset{\to 0}{a}_{j+1/2}(t) \cdot u_{j+1/2}(\bar{u})$$

$$u_{j+1/2} = \bar{u}_j + \varphi(\theta_j)\,(\bar{u}_{j+1} - \bar{u}_j)$$

where smoothness indicator

$$\theta_j = \frac{\bar{u}_j - \bar{u}_{j-1}}{\bar{u}_{j+1} - \bar{u}_j}$$

Observe:

$$\varphi(\theta) = \frac{1}{3} + \frac{\theta}{6} \quad \text{gives}$$

3rd order upwind biased!

$$u_{j+1/2} = \bar{u}_j + \frac{1}{3}(\bar{u}_{j+1} - \bar{u}_j) + \frac{1}{6}(\bar{u}_j - \bar{u}_{j-1}) =$$

$$= \frac{5}{6}\bar{u}_j + \frac{1}{3}\bar{u}_{j+1} - \frac{1}{6}\bar{u}_{j-1}$$

which is the same as what we wrote before

Use Koren limiter

smooth

$$\varphi = \max\left(0, \min\left(1, \underline{\underline{\frac{1}{3} + \frac{\theta}{6}}}, \theta\right)\right)$$

which is the equivalent of MC limiter for 3rd order upwind biased.

Must be combined with special "strong stability preserving" (SSP) RK3 integrator (see IMEX lecture)