Lecture 1 : Overview & Review of Fourier Analysis

Course webpage: https://math.nyu.edu/faculty/greengar/nm2/nm2.html

NYU Classes will be used for announcements, submitting homeworks, grades, etc.

Office hours: 7-8 pm Tuesdays, 9-10 am Fridays, or by appointment.
Grader's office hours TBD.

Main textbook freely available in PDF format from SIAM website using NYU network/proxy, as are several of the recommended texts
Many of the homework will be given with MATLAB templates in mind. You are free to use other languages but this may require some extra effort in translation…

Some material drawn from Prof. Donev's lectures:
http://cims.nyu.edu/~donev/Teaching/NMII/

Lecture 1: Overview & Review of Fourier Analysis

Please answer the following questions about background/interests and send to me via email as soon as possible.

1. Name, Degree Program, Year in Program, Thesis Advisor and topic (if known)
2. Previous academic degrees and relevant experience.
3. Brief statement about background in numerical analysis.
4. Topics in Numerical Methods II of particular interest.
5. Programming experience

There will be 5-6 homework assignments (50% of grade), posted on NYU Classes. (First assignment due Feb 16.)
Submit the solutions in PDF (with preference for write-ups in LaTex). If you are submitting multiple files, please merge all into a single zip file.
There will also be a take-home final/final project (50% of grade) due May 16.

Academic Integrity

If you use any external source (including Wikipedia), acknowledge the reference.

Discussing mathematical issues, algorithms, code design, etc. with classmates is encouraged.

However, your solutions and codes should be written individually, without duplicating/copying the work of other students (unless a specific homework is assigned as a group project - which may happen).

The final or final project should be carried out individually & without collaboration.

Please review the NYU academic integrity policy.

Estimating errors & convergence rates

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(LeVeque Appendix A)

For a target function $U(x)$, finite difference methods produce a set of values $U_i \approx U(x_i)$ at certain discrete points (equispaced or not).

How can we measure the error in this approximation? I.e. compare a set of discrete values with a function.

Let $e_i = U(x_i) - U_i$.

Is. $\|e\|_1 = \sum_{i=1}^{N} |e_i|$ a suitable measure of error? NO. Each e_i makes sense as the error at a particular point, but summing N such terms doesn't make sense. What we want is an estimate of $\|E(x)\|_1 = \int_a^b |E(x)|\,dx$. For equispaced data, this suggests $\|e\|_1 = h\sum_{i=1}^{N} |e_i|$. Similarly, for the 2-norm, $\|e\|_2 = \left(h\sum_{i=1}^{N} |e_i|^2\right)^{1/2}$. The maximum norm, however, is the same: $\|e\|_\infty = \max_i |e_i|$

Estimating errors & convergence rates

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(LeVeque Appendix A)

Let $E(h)$ denote the (scalar) error in the calculation with grid spacing h, typically some norm of the error over a grid.

$E(h) = \|U(h) - \hat{U}(h)\|$, where $\hat{U}(h)$ is the approximate solution.

The method is pth order accurate if $E(h) = Ch^p + o(h^p)$ as $h \to 0$.

We'll write $E(h) \approx Ch^p \quad \Rightarrow \quad E(h/2) = C(h/2)^p$.

Define $R(h) = E(h)/E(h/2) \quad \Rightarrow \quad R(h) \approx 2^p$ or $p = \log_2 R(h)$

Work with 2 grids (h_1, h_2): $E(h_1) \approx Ch_1^p, E(h_2) \approx Ch_2^p \qquad \Rightarrow p \approx \dfrac{\log(E(h_1)/E(h_2))}{\log(h_1/h_2)}$

What if you don't know the exact solution? For this, assume you've computed solution with $h, h/2, h/4$

$$\hat{E}(h) \equiv \hat{U}(h) - \hat{U}(h/2) \approx C\left(1 - \frac{1}{2^p}\right) h^p$$

$$\hat{E}(h/2) \equiv \hat{U}(h/2) - \hat{U}(h/4) \approx C\left(1 - \frac{1}{2^p}\right) \frac{h^p}{2^p}$$

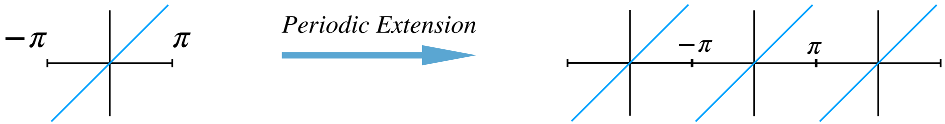
Thus $\hat{E}(h)/\hat{E}(h/2) \approx 2^p$.

Lecture 1 : Overview & Review of Fourier Analysis

Around 1800, Jean Baptiste Joseph Fourier was studying the PDEs governing heat flow and vibration, and hypothesized that any function could be represented by an infinite series of sines and cosines.

Mathematicians at the time found this highly implausible - but much of the modern world relies on this fact: signal processing, telecommunications, etc. grew from this observation. Moreover, much of modern mathematics came from trying to understand in what sense Fourier's claim was true.

We will not do this theory justice and will avoid most of the mathematical subtleties. That said, consider a periodic function $f(\xi)$ with period T: $f(\xi + L) = f(\xi)$. Setting $x = \xi \, \dfrac{2\pi}{L}$, we have $f(x + 2\pi) = f(x)$, so we will assume the period is 2π for convenience.

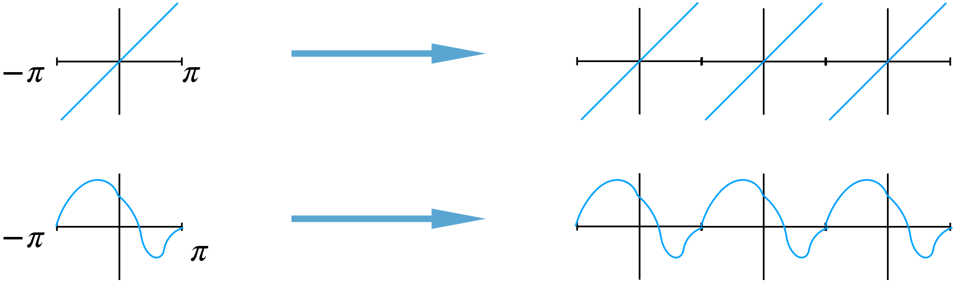


Peter Olver: Topics in Fourier Analysis:DFT & FFT, Wavelets, Laplace Transform

Cornelius Lanczos: Discourse on Fourier Series

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx)$$

1. Under what conditions does such a series converge?
2. For what functions is such a representation possible?
3. If it is possible, how do we determine α_n, β_n?
4. Can we integrate and differentiate the series?



$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx) \qquad (1)$$

Orthogonality relations with respect to L_2 inner product: $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\; g(x)\, dx$

$$\int_{-\pi}^{\pi} \cos(nx)\; \cos(mx)\, dx = \begin{cases} 2\pi,\; m = n = 0 \\ \pi, m = n \neq 0 \\ 0, m \neq n \end{cases} = \begin{cases} 2\pi,\; m = n = 0 \\ \pi\, \delta_{mn}, m = \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nx)\; \sin(mx)\, dx = \begin{cases} 0,\; m = n = 0 \\ \pi\, \delta_{mn}, m = \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\; \sin(mx)\, dx = 0$$

Assuming (1) is well-defined and that we can integrate term by term, we have

$$\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\; \cos(nx)\, dx, \beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\; \sin(nx)\, dx$$

With this choice of α_n, β_n, (1) is the Fourier series of $f(x)$.

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx) \qquad (1)$$

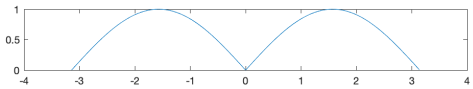
Example 1: $f(x) = |\sin x|$

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{0} |\sin x|\, dx + \frac{1}{\pi}\int_{0}^{\pi} |\sin x|\, dx = \frac{2}{\pi}\int_{0}^{\pi} |\sin x|\, dx = \frac{4}{\pi}$$

$$\alpha_n = \frac{2}{\pi}\int_{-\pi}^{0} |\sin x|\ \cos(nx)\, dx = \begin{cases} -\frac{4}{\pi}\frac{1}{n^2-1}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$\beta_n = 0$ Why?

$$|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nx)}{4n^2 - 1}$$



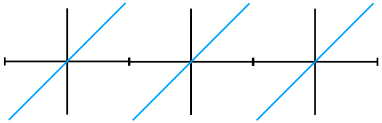
Demo

Example 2: $f(x) = x$

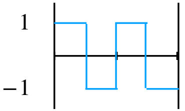
$\alpha_n = 0$ Why?

$$\beta_n = \frac{2}{\pi}\int_{-\pi}^{\pi} x\ \sin(nx)\, dx = 2\frac{(-1)^{n+1}}{n}$$

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n} = \begin{cases} x, & -\pi < x < \pi \\ 0, & x = \pm\pi\,. \end{cases}$$



Example 3: $f(x) = \text{Square wave}$

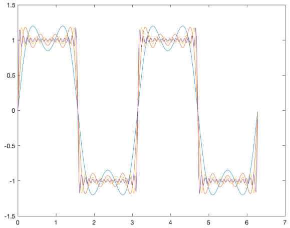


$0 \qquad 2\pi$

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\cos(nx)\,dx, \;\; \beta_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\sin(nx)\,dx$$

$$f_{SW}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}$$

Demo



Discontinuous, piecewise smooth

Jump discontinuities lead to convergence in L_2 but not uniform convergence. The error in L_∞ remains large near the singular point, and the overshoot/undershoot is about 9% of the magnitude of the jump (*Gibbs phenomenon or ringing*).

In many contexts, complex exponentials are simpler than real trigonometric functions:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}$$

Orthogonal basis, complex L_2 inner product: $\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ g^*(x)\,dx$

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \ldots = \left(\frac{1}{in}\right)^K \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(K)}(x)e^{-inx}dx$$

assuming $f \in C^K$. This estimate is not sharp (recall example 2)….

A function $f(z)$ defined on a strip $\{z : |Im(z)| < a\}, a > 0$ is 2π-periodic if $f(z + 2\pi) = f(z)$ in that strip

Theorem: If $f(z)$ is periodic and analytic in the strip, then for any ϵ, $|f_n| \le C(\epsilon)e^{(-a+\epsilon)|n|}$

Theorem: If $f(x) \in C^\infty$, then $|f_n|$ decays faster than any finite power of *n*.

} **"Spectrally accurate"**

Many possible symmetries: e.g. if $f(x)$ is real, then $f_{-n} = f_n^*$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}$$

The set of discrete Fourier coefficients $\{f_n\}$ is the **discrete spectrum**

The set $\{ |f_n|^2 \}$ is the **power spectrum**

$$\int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |f_n|^2$$

Why do we want to use a Fourier basis for representing functions?

Approximation/interpolation

Differentiation: $f^{(k)}(x) = \sum_{n=-\infty}^{\infty} (in)^k f_n e^{inx}$ Is this allowed?

Integration: $\int f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{f_n}{in} e^{inx} + C$ Is this allowed?

→

1. Need to sample $f(x)$
2. Need to be able to compute f_n
3. Need to understand accuracy

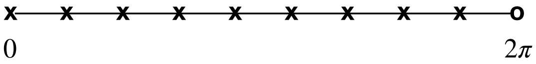
Solving initial-boundary value problems

Filtering: **DEMO**

$$f(x) \approx \sum_{n=-N/2}^{N/2-1} f_n e^{inx}$$ assuming N is even. [e.g. for N=8, we have: $\{f_{-4}, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3\}$].

Since $\{e^{inx}\}$ is an orthogonal basis for L_2, this is the best least-squares approximation in the span of the corresponding basis functions.

Now let's *discretize the interval:*



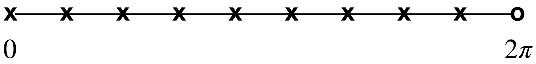
$$x_j = 2\pi j/N, \quad j = 0,\ldots,N-1$$

Let $\mathbf{f} \in \mathbb{C}^N$ with $\mathbf{f}_j = f(x_j)$ and

let $(\mathbf{f}, \mathbf{g}) = \sum_{j=0}^{N-1} f(x_j) g^*(x_j)$

Claim: Let $\Phi_n \in \mathbb{C}^N$ with $(\Phi_n)_j = e^{inx_j}$. Then $(\Phi_n, \Phi_m) = N\delta_{n,m}$.

(Discrete orthogonality)



$$x_j = 2\pi j/N, \quad j = 0,\ldots,N-1$$

Version 1

The Fourier interpolating polynomial is defined by $P_N(x) = \sum_{n=-N/2}^{N/2-1} \hat{f}_n e^{inx}$. Interpolation means that we want to solve the linear system

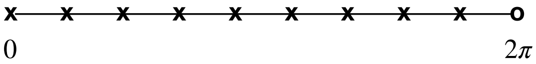
$$\sum_{n=-N/2}^{N/2-1} \hat{f}_n e^{inx_j} = f(x_j).$$

From orthogonality, we have $\hat{f}_n = \frac{1}{N}(\mathbf{f}, \Phi_n) = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j) e^{-inx_j}$. In linear algebraic notation, we have

$\hat{f}_n = \frac{1}{N}\mathscr{D}^*\mathbf{f}$, where $\mathscr{D}$ is the $N \times N$ matrix whose columns are given by $\begin{pmatrix} e^{inx_0} \\ e^{inx_1} \\ \cdots \\ e^{inx_{N-1}} \end{pmatrix}$.

BAD NEWS: Did you notice the indexing nightmare? Are you worried about the $-N/2$ mode?

GOOD NEWS: Because of orthogonality, we only have to apply $\mathscr{D}^* \Rightarrow O(N^2)$ work, not $O(N^3)$.



$$x_j = 2\pi j/N, \quad j = 0,\ldots,N-1$$

Version 2

Who cares about interpolating polynomials?

We're doing Fourier analysis, and already believe that $f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{-inx}dx$.

But, applying the trapezoidal rule, we have $f_n \approx \hat{f}_n = \frac{1}{2\pi}\sum_{j=0}^{N-1} f(x_j)e^{-inx_j}\, h = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j)e^{-inx_j}$, since $h = \frac{2\pi}{N}$.

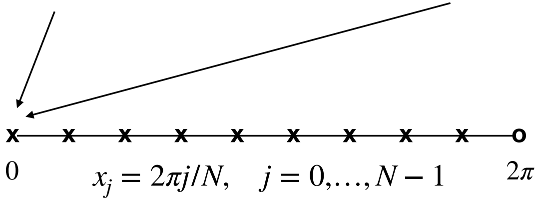
So - *the coefficients of the interpolating polynomial are approximations of the true Fourier coefficients.*
We still have work to do in *understanding accuracy* and in *making the approach fast*.

The Trapezoidal Rule is Spectrally Accurate for smooth, periodic functions

$$\int_a^b f(x)dx = h\left[\frac{f(a)}{2} + f(a+h) + \ldots + f(b-h) + \frac{f(b)}{2}\right] - \sum_{r=1}^{p-1} \frac{h^{r+1}B_{r+1}}{(r+1)!}[f^{(r)}(b) - f^{(r)}(a)] + O(h^{p+1})$$

assuming $f \in C^p$

Euler-McLaurin Formula



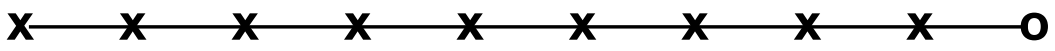
$x_j = 2\pi j/N, \quad j = 0,\ldots,N-1$

Thus (for smooth functions) not only is the Fourier series an extremely accurate and efficient approximation, it is computed to very high order accuracy using nothing more than the trapezoidal rule. For non-smooth functions, sampling at equispaced points and using our earlier formula still yields the interpolating trigonometric polynomial - but coefficients will no longer be good approximations of the true Fourier coefficients. [*They can still be computed, but with more complicated techniques*.]

Further reading: https://people.maths.ox.ac.uk/trefethen/publication/PDF/2014_150.pdf

The Trapezoidal Rule io Exact for trigonometric polynomials of degree $\leq N-1$

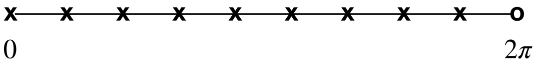
$$\int_a^b f(x)dx = h\left[f(a) + f(a+h) + \ldots + f(b-h)\right] \text{ for } f(x) = \sum_{n=0}^{N-1} a_n \cos(nx) + b_n \sin(nx)$$



$0 \qquad x_j = 2\pi j/N, \quad j = 0,\ldots,N-1 \qquad 2\pi$

1. What is a rule called if it integrates 2N functions exactly using only N nodes?
2. So, suppose you know f(x) is **band-limited** with band-limit N. How many points do you need on $[0,2\pi]$ to get the exact Fourier coefficients? (2N !)
3. Put differently, if a function is band-limited at frequency N, then 2N points are sufficient to represent (and interpolate) the function *exactly*. (That is the essence of the Shannon-Nyquist sampling theorem - which holds for the Fourier transform and functions on the real line as well.) In this case, $\hat{f}_n = f_n$.
4. What happens if you use 2N points but f(x) has frequency content beyond N modes? ***Aliasing***

Aliasing



Let $N = 2n + 1$

$$x_j = 2\pi j/N, \quad j = 0,\ldots,N-1$$

Recall the interpolating polynomial $P_N(x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$ and the truncated Fourier series $F_N(x) = \sum_{k=-n}^{n} f_k e^{ikx}$

Note that $e^{ikx_j} = e^{i(k+mN)x_j}$ so that we may write $f(x_j) = \sum_{k=-\infty}^{\infty} f_k \, e^{ikx_j} = \sum_{k=-n}^{n} \sum_{m=-\infty}^{\infty} f_{k+mN} \, e^{i(k+mN)x_j}$

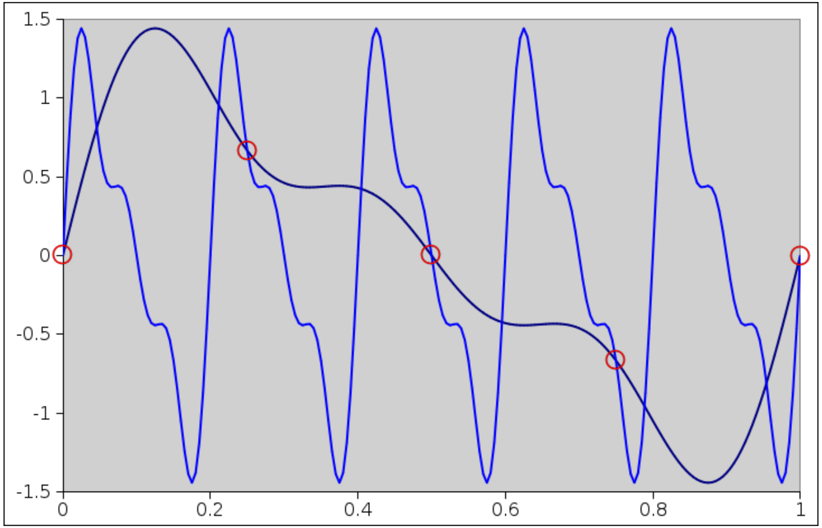
$$= \sum_{k=-n}^{n} \left(\sum_{m=-\infty}^{\infty} f_{k+mN} \right) e^{ikx_j}$$

$$\Rightarrow \hat{f}_k = \sum_{m=-\infty}^{\infty} f_{k+mN}$$

(Poisson Summation Formula)

$$|\hat{f}_k - f_k| \leq \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} |f_{k+mN}|$$

(Aliasing error)



1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1

Multiplication

Suppose that $f(x) = \sum_{k=-n}^{n} f_k e^{ikx}$ and that $g(x) = \sum_{k=-n}^{n} g_k e^{ikx}$.

$$N = 2n + 1$$

What is the Fourier series for $h(x) = f(x)\, g(x)$?

$$h(x) = \sum_{k=-n}^{n} f_k\, e^{ikx} \sum_{j=-n}^{n} g_j\, e^{ijx} = \sum_{k=-n}^{n} \sum_{j=-n}^{n} f_k\, g_j\, e^{i(j+k)x} = \sum_{m=-2n}^{2n} h_m\, e^{imx}$$

$$h_m = \sum_{k=-n}^{n} f_k\, g_{m-k}$$

$$m = -2n, \ldots, 2n$$

Linear convolution

The linear (aperiodic) convolution of two sequences of length N is a sequence of length 2N-1.

Sampling $h(x)$ with only N points can introduce **aliasing error**

https://www.mathworks.com/help/signal/ug/linear-and-circular-convolution.html

Periodic Convolution

Suppose that and that $f(x), g(x)$ are continuous $(2\pi)-$periodic functions. Their **periodic convolution** is $h(x) = (f * g)(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy.$

The Fourier series for $h(x)$ is $h_n = f_n\, g_n$. Proof (Exercise).

Thus, if you had a fast algorithm to compute f_n, g_n from $f(x), g(x)$ and a fast algorithm to compute $h(x)$ from $h_n = f_n\, g_n$, you can do fast periodic convolution.

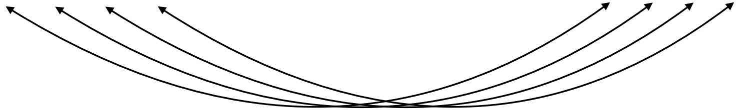
The Discrete Fourier transform (DFT)

$$f(x_j) = \sum_{n=-N/2}^{N/2-1} \hat{f}_n \, e^{inx_j} = \sum_{n=-N/2}^{N/2-1} \hat{f}_n \, e^{2\pi inj/N} \quad \text{for } j = 0, \ldots, N-1$$

"Fourier synthesis"

Re-indexing: Note that $e^{2\pi inj/N} = e^{2\pi ij} \, e^{2\pi inj/N} = e^{2\pi i(n+N)j/N}$

So, we can identify: $\{\hat{f}_{-4}, \hat{f}_{-3}, \hat{f}_{-2}, \hat{f}_{-1}, \hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3\}$ with $\{\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \hat{f}_6, \hat{f}_7\}$



Standard form for DFT: $f_j = \sum_{n=0}^{N-1} F_n \, e^{inx_j} = \sum_{n=0}^{N-1} F_n \, e^{2\pi inj/N} \quad \text{for } j = 0, \ldots, N-1$

The Fast Fourier transform (FFT)

Standard form for DFT: $f_j = = \sum_{n=0}^{N-1} F_n \, e^{2\pi i n j/N}$ for $j = 0,\ldots,N-1$ $\qquad N^2$ ops

Suppose $N = 2^K$. Then $f_j = \sum_{n=0}^{N-1} F_n \, e^{2\pi i n j/N} = \sum_{n=0}^{N/2-1} F_{2n} \, e^{2\pi i n j/(N/2)} + e^{2\pi i j/N} \sum_{n=0}^{N/2-1} F_{2n+1} \, e^{2\pi i n j/(N/2)}$ $\qquad (N^2/2) + N$ ops

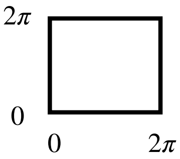
DFT of size *N*/2 on even terms F_{2n}

DFT of size *N*/2 on odd terms F_{2n+1}

Continuing this recursion for $\log_2 N$ levels yields an $O(N \log_2 N)$ algorithm.

Standard form for normalized adjoint DFT: $F_n = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \, e^{inx_j} = \frac{1}{N} \sum_{j=0}^{N-1} f_j \, e^{2\pi i n j/N}$ for $n = 0,\ldots,N-1$

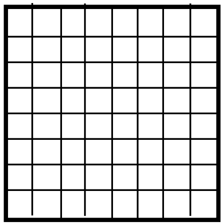
Periodicity in Higher Dimensions



$$f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{n,m} e^{i(nx+my)}$$

$$f_{n,m} \approx \hat{f}_{n,m} = \left(\frac{1}{2\pi}\right)^2 \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} f(x_j, y_k) e^{-i(nx_j+my_k)} h^2 = \frac{1}{N_x N_y} \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} f(x_j, y_k) e^{-2\pi inj} e^{-2\pi imk}$$

$$= \frac{1}{N_x} \sum_{j=0}^{N_x-1} e^{-2\pi inj} \left[\frac{1}{N_y} \sum_{k=0}^{N_y-1} f(x_j, y_k) e^{-2\pi imk} \right]$$



For nonperiodic functions, Fourier series are no longer applicable and we will need to make use of the Fourier transform:

$$\hat{f}(s) = \mathscr{F}[f](s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx}dx \qquad f(x) = \mathscr{F}^{-1}[\hat{f}](x) = \int_{-\infty}^{\infty} \hat{f}(s)e^{2\pi isx}ds$$

↓

Frequency domain

As with complex Fourier series, there are many symmetries possible::
e.g. if $f(x)$ is real, then $\hat{f}(s) = \hat{f}(-s)^*$. If $f(x)$ is even or odd, so is $\hat{f}(s)$.
$\mathscr{F}[f^{(K)}](s) = (2\pi is)^K \hat{f}(s)$

Convolution theorem: If $h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)dy$, then $\hat{h}(s) = \hat{f}(s) \cdot \hat{g}(s)$

Summary 1

Periodic functions are naturally represented in terms of trigonometric functions (or complex exponentials). This permits easy differentiation, integration and interpolation and filtering.

The convergence rate is determined by the smoothness of the function (including its periodic extension). If $f(x) \in C^\infty$ (or analytic), the convergence rate is _spectral_ or _superalgebraic._

If $f(x) \in C^K$, then the truncated Fourier approximation converges at a rate of the order $o\left(\frac{1}{N^K}\right)$.

Once $f(x)$ is continuously differentiable, the convergence is uniform.

If $f(x)$ is discontinuous, but piecewise smooth, the Fourier series converses in L_2 but not uniformly (Gibbs phenomenon). Much of modern analysis grew out of understanding this convergence theory.

- Carleson's Theorem (1966!): The Fourier expansion of any function in L^2 converges almost everywhere
- Kahane, Katznelson (1965): For any given set *E* of measure zero, there exists a continuous function f such that the Fourier series of f fails to converge on any point of *E*.

In numerical analysis, these considerations are secondary. What we require is tools that:
(a) identify the structure of the solution, (b) monitor the convergence process, (c) estimate the error and (d) compute the solution quickly and without user intervention.

Smooth functions on an interval

Without periodicity, smooth functions are naturally (and historically) represented in a polynomial (**or piecewise polynomial**) basis. This topic was covered in Numerical Methods 1 (also, Appendix B in LeVeque's text).

If $w(x)$ is a weight function on an interval $[a, b]$ (positive, integrable, etc.), it induces an inner product:

$$(f, g) = \int_a^b f(x)\, g(x)\, w(x)\, dx$$

and a sequence of orthogonal polynomials $\mathscr{P}_0, \mathscr{P}_1, \ldots, \mathscr{P}_n, \ldots$ of increasing degree with $(\mathscr{P}_n, \mathscr{P}_m) = 0$ for $m \neq n$.

In this class, we will care most about Chebyshev and Legendre polynomials:

Legendre: $[a, b] = [-1,1]$ and $w(x) = 1$.

$$(n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)\,.$$

$$P_0 = 1,\ P_1 = x,\ P_2 = \frac{3}{2}x^2 - \frac{1}{2},\ P_3 = \frac{5}{2}x^3 - \frac{3}{2}x, \ldots$$

Chebyshev: $[a, b] = [-1,1]$ and $w(x) = 1/\sqrt{1 - x^2}$.

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)\,.$$

$$T_0 = 1,\ T_1 = x,\ T_2 = 2x^2 - 1,\ T_3 = 4x^3 - 3x, \ldots$$

$$T_m(\cos\theta) = \cos(m\theta)\,.$$

Smooth functions on an interval

Why are the Legendre and Chebyshev bases good for smooth functions?

Suppose we let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ or $f(x) = \sum_{n=0}^{\infty} c_n T_n(x)$ on the interval $[-1,1]$ and that $f(x) \in C^K$.

Then $a_n, c_n = O\left(\frac{1}{n^K}\right)$ - *just like for the Fourier series approximation of a smooth periodic function.*

And the aproximation error is $O\left(\frac{1}{n^{K-1}}\right)$ *or better.* [Homework 1].

And again: If $f(x) \in C^{\infty}$ (or analytic), the convergence rate is *spectral* or *superalgebraic.*

Approximation/interpolation
Differentiation:
Integration:
Solving initial-boundary value problems
Filtering

→

1. Need to sample $f(x)$
2. Need to be able to compute a_n, c_n

Clenshaw-Curtis Quadrature

Recall that the trapezoidal rule with N points integrates 2N-1 trigonometric functions exactly.

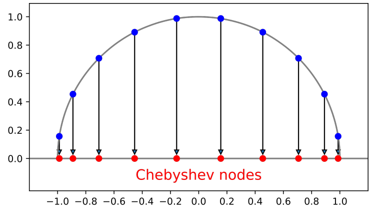
Using a Chebyshev basis $f(x) = \frac{1}{2} c_0 T_0(x) + \ c_1 T_1(x) + \ \dots + \ c_n T_n(x) + \dots$ and the inner product

$(f, g) = \int_{-1}^{1} f(x)\, g(x)\, (1 - x^2)^{-1/2}\, dx,$

we have $c_k = \frac{2}{\pi} \int_{-1}^{1} f(x)\, T_k(x)\, (1 - x^2)^{-1/2}\, dx = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta)\ \cos(k\theta)\, d\theta\,.$

Let's put an equispace grid on the latter interval $[0,\pi]$: $\theta_n = n\pi/N$ for $n = 0,\dots,N$. Let's try:

$$c_k \approx \frac{2}{N} \left(\frac{f(1)}{2} + \frac{f(-1)}{2} (-1)^k + \sum_{n=1}^{N-1} f(\cos\theta_n) \cos(nk\pi/N) \right)$$



This is a spectrally accurate quadrature scheme (Why?) and can be computed using the FFT.

One can also use the "classical" Chebyshev nodes: $\theta_n = \frac{(n + \frac{1}{2})\pi}{N}$ for $n = 0,\dots,N-1$.

Gaussian Quadrature

Using a Legendre basis $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ it can be shown that. $(P_n, P_m) = \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\delta_{nm}$.

We have $a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\, P_k(x)\, dx$

We'd like a spectrally accurate quadrature scheme for this but have no recourse to periodic tricks.

Thus, we want a rule of the form: $\int_{-1}^{1} f(x)\, dx = \sum_{n=1}^{N} w_n f(x_n)$ to be exact for polynomials of degree up to 2N-1.Gauss showed that this can be accomplished by choosing x_n as the roots of $P_N(x)$ and the weights

$$w_n = \frac{2}{(1-x_n^2)[P_N'(x_n)]^2}\ , \qquad E_n = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n!)]^3} f^{(2n)}(\xi)$$

GOOD NEWS: There are *recent* fast algorithms for computing $\{x_n, w_n\}$ in linear time.

BAD NEWS: A fast transform for $a_k = \frac{2k+1}{2}\sum_{n=1}^{N} f(x_n)\, P_k(x_n) w_n$ is more involved than the FFT or FCT.

What are possible reasons for wanting to use the Legendre basis instead of the Chebyshev basis?

Adaptive Quadrature

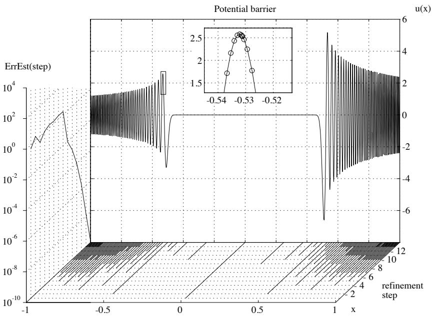


FIG. 5. *The adaptive solution of the potential barrier problem (Example 4).*

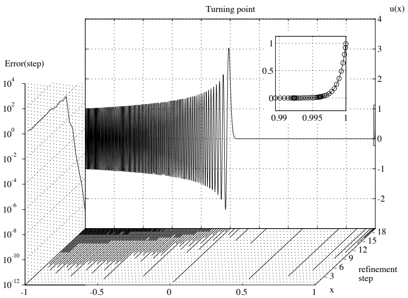


FIG. 4. *The adaptive solution of a turning point problem (Example 3). In this case, the magnified window examines the boundary layer at $x = 1$.*

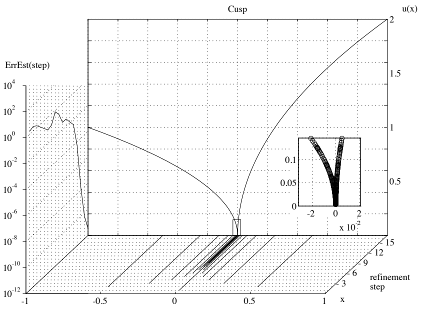
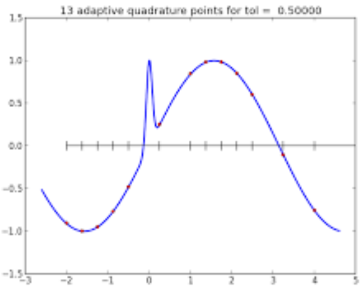
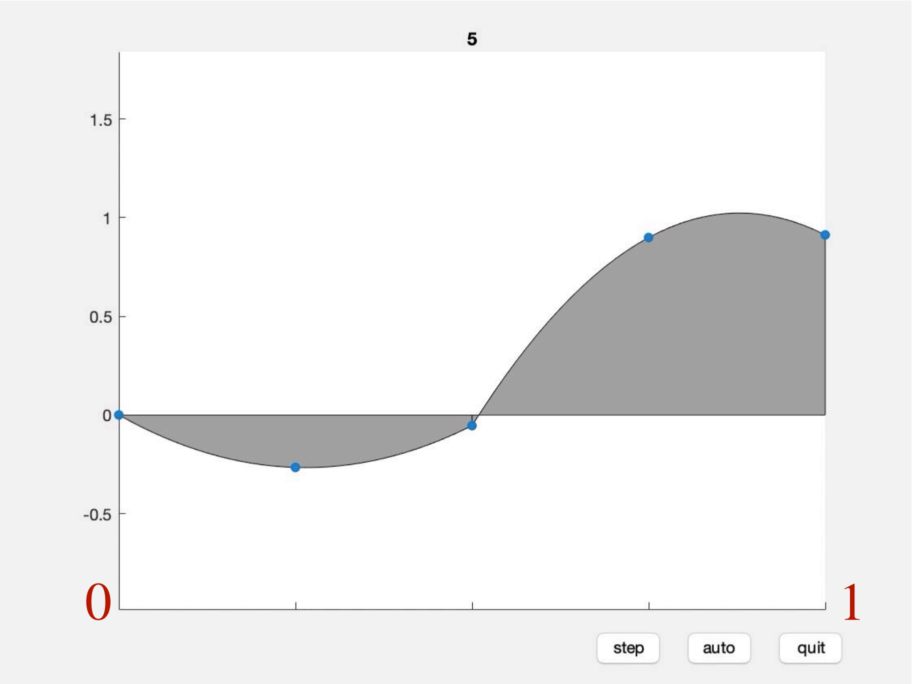


FIG. 6. *The adaptive solution of the cusp problem (Example 5).*



-> SKETCH

Adaptive quadrature



f= @(x) -sqrt(x)*log(x)+sin(20*x)
a= 10^(-8); b = pi;
q= quadgui(f,a,b)

Boundary Value Problems

Periodic boundary value problems: Constant coefficient vs variable coefficient problems.

Spectral vs *pseudospectral* methods. Setting up the linear system.