STOKES FLOW

CFD, SPRING 2013, A. DONEV

For viscous-dominated flows (small Reynolds numbers), it is often (though not always) the case that the inertial terms $\rho(\partial_t v + v \cdot \nabla v)$ play little to no role and can be neglected. The fluid flow instantaneously relaxes to a steady state described by the

steady Stokes equation:

$$\begin{cases} \nabla p = \mu \nabla^2 v + f(r,t) \\ \nabla \cdot v = 0 \end{cases}$$

← elliptic problem

Since this is a linear equation, it can be solved via the method of Green's functions.

Let us first consider an unbounded domain, with velocity decaying like $1/r$ and pressure like $1/r^2$ at infinity (free-space Green's functions)

The solution to ③

$$\begin{cases} \nabla p = \mu \nabla^2 \vec{v} + \vec{f}\delta(\mathbf{x}_0) \\ \nabla \cdot \vec{v} = 0 \end{cases}$$ ← Dirac δ function

, with decay

is the Oseen tensor

$$\begin{cases} v_i(x) = \frac{1}{8\pi\mu} S_{ij}(x, x_0) f_j \\ p(x) = \frac{1}{8\pi} P_j f_j \end{cases}$$

Oseen tensor

where

$$\boxed{S_{ij}(x, x_0) = \frac{\delta_{ij}}{\|x - x_0\|} + \frac{(x_i - x_{0,i})(x_j - x_{0,j})}{\|x - x_0\|^3}}$$

$$P_j(x, x_0) = \frac{2(x_j - x_{0,j})}{r^3}$$

STOKESLET

The stress tensor is (4)

$$\sigma_{ik}(x) = \mu\left(\frac{\partial v_k}{\partial r_i} + \frac{\partial v_i}{\partial r_k}\right) = \frac{1}{8\pi} T_{ijk}(x, x_0) f_j$$

$$T_{ijk} = -\frac{6(x_i - x_{0,i})(x_j - x_{j,0})(x_k - x_{0,k})}{\|x - x_0\|^5}$$

All of these can be obtained from the Green's function of the Laplace equation $\nabla^2 \varphi = 0$ and the bi-harmonic equation $\nabla^4 \psi = 0$.

Note that <u>all</u> are <u>singular</u> as $x \to x_0$

One can obtain a non-singular fundamental solution by <u>regularizing</u> the δ function:

$$\delta(x) \longrightarrow \delta_\varepsilon(x) \quad \text{where} \quad \int \delta_\varepsilon(x)\,dx = 1$$

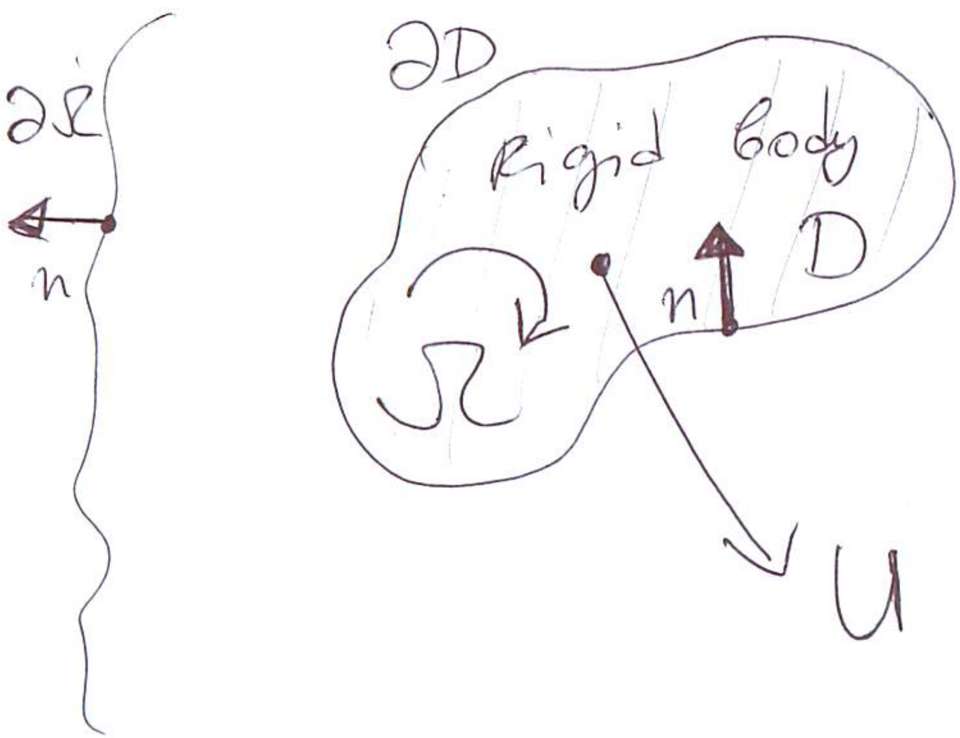
$$\begin{cases} \nabla p = \mu \nabla^2 v + f\,\delta_\varepsilon(x_0) \\ \nabla \cdot v = 0 \end{cases}$$

$$\Rightarrow \quad v_i(x) = \frac{1}{8\pi\mu} S^{\varepsilon}_{ij}(x, x_0)\, f_j$$

<u>REGULARIZED STOKESLET</u>

For explicit formulae see
CORTEZ, FAUCI, MEDOVIKOV, 2005

From these free-space Green's functions, we can now build solutions in bounded domains.



fluid domain

Ω

(say unbounded for simplicity)

From Green's third identity one can obtain the following (7)

Boundary Integral Formulation:

$x_0 \notin \partial D$, $x_0 \in \Omega$

$$u_j(x_0) = -\frac{1}{8\pi\mu} \int_{\partial D} S_{ij}(x, x_0) f_i(x) \, ds(x) - \frac{1}{8\pi} \int_{\partial D} u_i(x) T_{ijk}(x, x_0) n_k \, ds(x)$$

Integral over surface only

normal to surface

$$\boxed{f_i = -\sigma_{ik} n_k}$$ ← boundary traction (force) on rigid body

Now we consider limit ⑧

$$x_0 \to \partial\Omega$$

It is, perhaps, easier to consider a regularized version of the Green's identity (see Cortez et al.):

$$\int_\Omega u_j(x)\,\delta_\varepsilon(x-x_0)\,dV(x) = \qquad x_0 \in \Omega$$

including ∂D

(non-singular)

$$-\frac{1}{8\pi\mu}\int_{\partial D} S_{ij}^{\varepsilon}(x,x_0)\,f_i(x)\,ds(x)$$

$$-\frac{1}{8\pi}\int_{\partial D} u_i(x)\,T_{ijk}^{\varepsilon}(x,x_0)\,n_k\,ds(x)$$

(9)

As $\varepsilon \to 0$ and $x_0 \in \partial\Omega$ the integral is over half:

$$\int_\Omega u_j(x)\, \delta_\varepsilon(x-x_0)\, dV(x) \to \frac{1}{2} u_j(x_0)$$

for $x_0 \in \partial\Omega$ due to continuity of velocity (but pressure is <u>not</u> continuous)

$$\Rightarrow \quad (*) \quad \boxed{\frac{1}{2} u_j(x_0) = -\frac{1}{8\pi} \int_{\partial D} \left[\mu^{-1} S_{ij}(x,x_0) f_i(x) + u_i(x) T_{ijk}(x,x_0) n_k \right] dS(x) \qquad x_0 \in \partial D}$$

Integral equation of
$\begin{cases} 2^{nd} \text{ kind for } u \text{ (given } f) \\ 1^{st} \text{ kind for } f \text{ (given } u(\partial\Omega)) \end{cases}$

If we know the velocity on the boundary of the body; e.g.; (10)

$$\boxed{u(x_0 \in \partial\Omega) = U + \Omega \times (X_0 - X_{cm})}$$

Rigid-body motion

velocity of body

angular velocity

center-of-mass

then (*) is an integral equation of the first kind for the surface traction (force density on surface) $f(x_0 \in \partial\Omega)$ and This equation is ill-conditioned and there are techniques to convert it to a second kind equation.

Once we know $f(x \in \partial\Omega)$, we (11)
can evaluate $u(x \in \Omega)$ via the
Green's identity.

This is the basis for the
<u>boundary-integral method</u>

The details are, however, very
<u>technical</u>: How to represent the
unknown surface densities (conditioning),
how to discretize the singular
integrals (quadrature), how to solve
the linear systems, etc.

See Leslie Greengard & Mike Shelley
for reading

Here we will consider a simple (12)
approach of regularized Stokeslets,
which is also closely-related to
the immersed boundary method, as

we will see.
First, just like in the IB method,
let's extend the velocity field into
the body as well, i.e., pretend there
is "fluid" everywhere.

Inside D, $v(r,t)$ must be
a rigid-body motion

If

$$u(x_0 \in \partial\Omega) = U + \Omega \times (x_0 - x_{cm})$$

then

$$\boxed{u(x \in D) = U + \Omega \times (x - x_{cm})} \quad (**)$$

solves the stokes equations with

$$\boxed{p = \text{const} \quad \text{for } x \in D} \quad (\nabla p = 0)$$

$\nabla \cdot u = 0$ and

$$\boxed{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0 \quad \text{inside } D}$$

zero-deformation condition
(equivalent to *)

So if we were to solve the Stokes equation everywhere, inside the rigid body it would be consistent with the rigid-body motion, i.e., the "fluid" velocity would match the solid velocity. (note that Peskin's approach for an elastic body leads to similar conclusions)

From (**) it can be shown that

$$\frac{1}{8\pi}\int_{\partial D} u_i(x)\, T^{\varepsilon}_{ijk}(x, x_0)\, n_k \, ds(x) = \int_{D} u_j\, \delta_{\varepsilon}(x - x_0)\, dV(x)$$

body only!

If we now plug this into the boundary integral equation on page 8,

$$\int_{R^3} u_j(x)\, \delta_\varepsilon (x-x_0)\, dV(x) = -\frac{1}{8\pi\mu} \int_{\partial D} S^{\varepsilon}_{ij}(x, x_0) f_i(x) ds$$

R^3 whole domain

$\delta_\varepsilon (x-x_0) \downarrow u_j(x_0)$ as $\varepsilon \to 0$ even if $x_0 \in \partial\Omega$

Let us denote

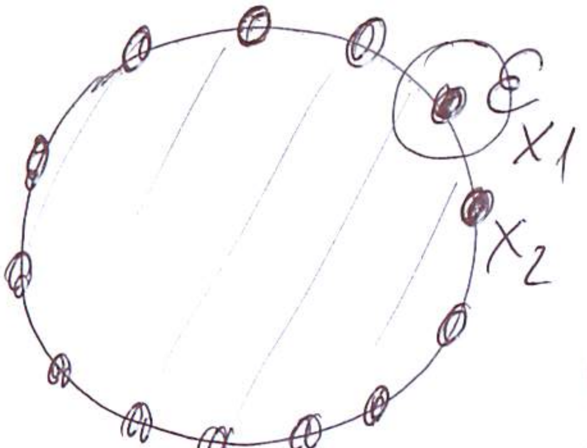
$$V(x_0) = \int_{R^3} u(x)\, \delta_\varepsilon (x-x_0)\, dV(x)$$

where $V(x_0) \simeq u(x_0)$ if $\varepsilon \to 0$ and $u(x)$ is continuous

$$\boxed{V_j(x_0) = -\frac{1}{8\pi\mu} \int_{\partial D} S_{ij}^{\varepsilon}(x, x_0)\, f_i(x)\, ds(x)}$$

$x_0 \in \mathbb{R}^3$, even on boundary ∂D

Discretize integral by simple sum over discrete <u>markers</u> $x_1, \dots, x_N$



This is just like the immersed boundary method, but here ε is not related to a fluid grid (there is no fluid grid!), but the distance between markers.

$$\boxed{\underset{\substack{\uparrow\\ \text{velocity of marker}}}{V}(x_i) = -\frac{1}{8\pi\mu}\sum_{k=1}^{N} \overset{\leftrightarrow}{S}^{\varepsilon}(x_k, x_i)\, F_k}$$

which is a linear system of equations for the forces on every marker (regularized Stokeslets), which can be, in principle, solved by GMRES (notice that it is not only singular but also ill-conditioned.....)

Let's compare this method of regularized Stokeslets to the immersed boundary method

If we used IB for Stokes flow, we would solve:

$$\begin{cases} \nabla p = \mu \nabla^2 v + \sum_{k=1}^{N} F_k \, \delta_h(r - X_k) \\ \nabla \cdot v = 0 \end{cases}$$

and then set

$$V_k = \frac{\partial X_k}{\partial t} \approx \int v(r) \, \delta_h(r - X_k) \, dr$$

quadrature over grid

which is exactly what the method of regularized stokeslets approximates the velocity at X_k with!

The key difference between Stokeslets and markers in IB method is that in IB we use a fluid solver (on a grid) to solve NS or Stokes equations, while for boundary-integral or regularized Stokeslets we use analytical solutions of the Stokes equation.

The main advantage of IB is its generality and flexibility. But the cost is more computational cost and grid artifacts