Spring 2021: Numerical Analysis
Assignment 1 (due Feb. 22, 2021 at 2pm EST)

Read homework instructions on class homepage carefully before submitting on Gradescope!

1. **[5pt]** We search for solutions in $[1, 2]$ to the equation

$$x^3 - 3x^2 + 3 = 0.$$

   (a) Compute the first iterates $x_0, \dots, x_5$ of the secant method in $[1, 2]$.

   (b) Compute the first iterates $x_0, \dots, x_5$ using Newton's method with starting value $x_0 = 1.5$.

   (c) Compute the first iterates using Newton's method with starting value $x_0 = 2.1$. Sketch the equation graph and try to explain the behavior.

2. **[5pt]** In this problem, you will write a code that will find all of the roots of the function $f(x) = x^5 - 3x^2 + 1$ inside the interval $[-2, 2]$. For each part, please submit an answer that includes your reasoning, your code, and the values you obtain for the roots.

   (a) Using the fact that the roots of f on this interval are separated by at least 0.25, write a program that implements the bisection algorithm to obtain estimates of the roots that are accurate to within 0.1.

   (b) Using the estimates of the roots obtained in the previous part, write a program that implements Newton's method to refine the values of the roots such that the difference between successive iterates is at most 10^{-12}.

   (c) How confident are you that you found the roots in part (b)? How accurate do you think they are? [*Hint: Plotting the function is always a good idea. Also, while bisection is slow, you can use it to obtain reasonably accurate estimates of the roots and compare. Lastly, maybe Matlab has some function that can help you get the roots?*]

3. **[10pts]** In this problem, you will prove the rate of convergence for the secant method.

   (a) Show that the secant method

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

   can be rewritten in the form:

$$x_{k+1} = \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)}. \tag{1}$$

   (b) Now, denote the root of f to be ξ, so that $f(\xi) = 0$. Also assume that f is twice continuously differentiable and that $f' > 0$ and $f'' > 0$ in a neighborhood of ξ. Define the quantity ψ to be:

$$\psi(x_k, x_{k-1}) = \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)},$$

   where x_{k+1} is as in (1). Compute (for fixed value of x_{k-1})

$$\varphi(x_{k-1}) = \lim_{x_k \to \xi} \psi(x_k, x_{k-1}).$$

(c) Now compute

$$\lim_{x_{k-1}\to\xi} \varphi(x_{k-1}),$$

and therefore show that

$$\lim_{x_k, x_{k-1}\to\xi} \psi(x_k, x_{k-1}) = \frac{f''(\xi)}{2f'(\xi)}.$$

(d) Next, assume that the secant method has convergence order q, that is to say that

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|^q} = A < \infty.$$

Using the above results, show that $q - 1 - 1/q = 0$, and therefore that $q = (1+\sqrt{5})/2$.

(e) Finally, show that this implies that

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|^q} = \left(\frac{f''(\xi)}{2f'(\xi)}\right)^{q/(1+q)}.$$

4. **[5pt]** Raytracing is an algorithm that involves finding the point at which a ray (a line with a direction and an origin) intersects a curve or surface. We will consider a ray intersecting with an ellipse. The general equation for an ellipse is

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 - 1 = 0$$

and the equation for a ray starting from the point $P_0 = [x_0, y_0]$ in the direction $\mathbf{V}_0 = [u_0, v_0]$, is

$$\mathbf{R}(t) = [x_0 + tu_0, y_0 + tv_0]$$

where $t \in [0, \infty)$ parameterizes the ray. In this problem we will take $\alpha = 3$, $\beta = 2$, $P_0 = [0, \beta]$, $\mathbf{V}_0 = [1, -0.3]$. In this problem you will write a code which computes the intersection of the given ray and the ellipse and plot your results.

(a) Plug the equation for the ray, $\mathbf{R}(t)$, into the equation for the ellipse and analytically (with pen and paper) solve for the value of t which gives the point of intersection, call it t_i.

(b) Perform the same calculation numerically using your favorite root finder. Report your answer to within an error (either absolute or relative) of 10^{-6} and justify how you found the minimum number of iterations required to achieve this tolerance. Also report the point of intersection $P_i = \mathbf{R}(t_i)$.

5. **[Optional fun problem, continues problem 4]**
If the walls of the ellipse are perfect mirrors, a ray of light will reflect infinitely around within the ellipse. We will write an algorithm to compute it's trajectory. Using the same parameters as the previous problem. Implement the following algorithm for 50 steps.

At step k of the process, we are given a point on the ellipse $P_k = [x_k, y_k]$ and a ray direction $\mathbf{V}_k = [u_k, v_k]$

(a) Using P_k and $\mathbf{V}_k$ and your favorite root finder, calculate the value of t corresponding to the point of intersection call it t_i^k.

(b) Compute $P_{k+1} = \mathbf{R}(t_i^k)$.

(c) Compute the normal vector at $P_{k+1} = [x_{k+1}, y_{k+1}]$ as $\mathbf{N}_{k+1} = [\frac{b}{a}x_{k+1}, \frac{a}{b}y_{k+1}]$. Using this compute the *normalized* (unit) normal $\mathbf{n}_{k+1} = \mathbf{N}_{k+1}/||\mathbf{N}_{k+1}||_2$.

(d) Compute the unit ray vector $\mathbf{w}_k = \mathbf{V}_k/||\mathbf{V}_k||_2$ and update the new ray vector using the reflection formula

$$\mathbf{V}_{k+1} = \mathbf{w}_k - 2\left(\mathbf{w}_k \cdot \mathbf{n}_{k+1}\right)\mathbf{n}_{k+1}$$

(e) repeat from step 1 using $\mathbf{V}_{k+1}$ and P_{k+1}.

Plot the trajectory (all of the points that you computed with line connecting them) as well as the ellipse. Also report the 50th point of intersection. Note that the shape that encloses all of the reflected rays is called a "caustic," see figure below for an illustration.