Spring 2021: Numerical Analysis
Assignment 4 (due April 7th 2pm EST)

1. **[Eigenvalue/vector properties, 9pts]** Prove the following statements, using the basic definition of eigenvalues and eigenvectors, or give a counterexample showing the statement is not true. Assume $A \in \mathbb{R}^{n \times n}$, $n \geq 1$.

   (a) [1pt] If λ is an eigenvalue of A and $c \in \mathbb{R}$, then $\lambda + c$ is an eigenvalue of $A + cI$, where I is the identity matrix.

   (b) [1pt] If λ is an eigenvalue of A and $c \in \mathbb{R}$, then $c\lambda$ is an eigenvalue of cA.

   (c) [1pt] If λ is an eigenvalue of A, then for any positive integer p, λ^p is an eigenvalue of A^p.

   (d) [1pt] Every matrix with $n \geq 2$ has at least two *distinct* eigenvalues.

   (e) [1pts] Every real matrix has a real eigenvalue.

   (f) [1pt] If A is singular, then it has an eigenvalue equal to zero.

   (g) [1pt] If all the eigenvalues of a matrix A are zero, then $A = 0$.

   (h) [2pts] If $\widetilde{A}$ is "similar" to A, which means that there is a nonsingular matrix P such that $A = P\widetilde{A}P^{-1}$, then if λ is an eigenvalue of A, it is also an eigenvalue of $\widetilde{A}$. How do the eigenvectors of $\widetilde{A}$ relate to the eigenvectors of A?

2. **[Roots of polynomials, 8 pts]** An efficient way to find individual roots of a polynomial is to use Newton's method. However, as we have seen, Newton's method requires an initialization close to the root one wants to find, and it can be difficult to find *all* roots of a polynomial. Luckily, one can use the relation between eigenvalues and polynomial roots to find all roots of a given polynomial. Let us consider a polynomial of degree n with leading coefficient 1:

$$p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n \quad \text{with } a_i \in \mathbb{R}.$$

   (a) [3pts] Show that $p(x)$ is the characteristic polynomial of the matrix (sometimes called a companion matrix for p)

$$A_p := \begin{bmatrix} 0 & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

   Thus, the roots of $p(x)$ can be computed as the eigenvalues of A_p using the QR algorithm (as implemented, e.g., in MATLAB's `eig` function).

   (b) [5pts] Let us consider Wilkinson's polynomial $p_w(x)$ of order 15, i.e., a polynomial with the roots $1, 2, \ldots, 15$:

$$p_w(x) = (x-1) \cdot (x-2) \cdot \ldots \cdot (x-15).$$

   The corresponding coefficients can be found using the `poly()` function. Use these coefficients in the matrix A_p to find the original roots again, and compute their *relative* error. Compare with the build-in method (called `roots()`) for finding the roots of a polynomial.[1]

[1] Note that for MATLAB functions that do not use external libraries, you can see how they are implemented by typing `edit name_of_function`. Doing that for the `roots` function will show you that MATLAB implements root finding exactly using a companion matrix as described above.

3. **[Power Method, 8pts]**

   (a) [3pts] Implement the Power Method for an arbitrary matrix $A \in \mathbb{R}^{n \times n}$ and an initial vector $\boldsymbol{x}_0 \in \mathbb{R}^n$, and test that it works (i.e., validate it) in some way. [*Hint: Try applying it to a diagonal matrix, for example.*]

   (b) [5pts] Use your code to find an eigenvector of

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix},$$

   starting with $x_0 = (1, 2, -1)^T$ and $x_0 = (1, 2, 1)^T$. Report the 6th iterate for each of the two initial vectors. Then use MATLAB's `eig(A)` to examine the eigenvalues and eigenvectors of A. Do the two sequences converge to the expected eigenvector according to the theory covered in class — if not why not?

4. **[Square root of matrix, 12pts]** In homework 2 you considered applying Newton's method to compute a matrix square root. The eigenvalue decomposition can be used to generalize scalar functions to matrix valued functions. For this problem it is OK to assume that $\boldsymbol{A}$ is normal or that it is Hermitian/symmetric, although some parts can be proven by only assuming $\boldsymbol{A}$ is non-defective.

   (a) [1pt] Relate the eigenvalue decomposition of $\boldsymbol{A}^{-1}$ to that of $\boldsymbol{A}$.

   (b) [3pts] Given a square matrix $\boldsymbol{A}$ and real number t, the *matrix exponential* $e^{t\boldsymbol{A}}$ is defined via the Taylor series for the exponential function:

$$e^{t\boldsymbol{A}} = 1 + t\boldsymbol{A} + \frac{(t\boldsymbol{A})^2}{2!} + \frac{(t\boldsymbol{A})^3}{3!} + \frac{(t\boldsymbol{A})^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{(t\boldsymbol{A})^k}{k!} \tag{1}$$

   Can you relate the eigenvalue decomposition of $e^{t\boldsymbol{A}}$ (as defined above) with that of $\boldsymbol{A}$?

   (c) [4pts] The square root of a Hermitian/symmetric positive semidefinite matrix $\boldsymbol{A}$ (i.e., a matrix with all *non-negative* eigenvalues) is a matrix $\boldsymbol{X}$ such that $\boldsymbol{X}^*\boldsymbol{X} = \boldsymbol{A}$, where star denotes complex conjugate transpose. If you are given the eigenvalue decomposition of $\boldsymbol{A}$, can you find at least one such matrix $\boldsymbol{X}$?
   Often the matrix $\boldsymbol{X}$ is restricted to be Hermitian also (i.e., $\boldsymbol{X}^2 = \boldsymbol{A}$). Can you find a Hermitian square root of $\boldsymbol{A}$ from the eigenvalue decomposition of $\boldsymbol{A}$? Is such a square matrix unique? If not, can you think of some way to make it unique (just like we decide $\sqrt{4} = 2$ be unique instead of the more general $\sqrt{4} = \pm 2$)?

   (d) [1pts] Using what you found above, can you think of how you may apply any scalar function $f(x)$ (say sine) to a symmetric matrix?

   (e) [3pts] Find a symmetric square root of the matrix (explain how) using Matlab

$$\boldsymbol{A} = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 1 & 2 & 4 & 8 \end{pmatrix}. \tag{2}$$

   Do *not* use built-in functions for square roots of matrices (but it is OK to use them to validate your answer).