Spring 2021: Numerical Analysis
Assignment 6: Polynomial approximation and quadrature
Due May 10th 2pm EST

1. **[Connection between interpolation and 2-norm approximation, 10pts]** Using a set of disjoint points (nodes) $x_0, \dots, x_n$ in $[a, b]$, we define an inner product for polynomials $p(x), q(x)$ as
$$\langle p, q \rangle := \sum_{i=0}^{n} p(x_i)q(x_i).$$
   (a) [3 pts] This is an inner product for each $\mathcal{P}_k$ with $k \le n$, where $\mathcal{P}_k$ denotes the space of polynomials of degree k or less. Why is $\langle \cdot, \cdot \rangle$ not an inner product for $k > n$?
   (b) [3 pts] Show that the Lagrange polynomials $L_k(x)$ corresponding to the nodes $x_0, \dots, x_n$ are orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$.
   (c) [4 pts] For a continuous function $f : [a, b] \to \mathbb{R}$, compute its optimal approximation in $\mathcal{P}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle$ and compare with the polynomial interpolant of f on the same set of nodes.

2. [**Quadratic approximation of sine, 6pts**] Compute explicitly the optimal L_2 quadratic approximation of $\sin(x)$ on $[0, \pi]$ for the standard L_2 inner product. Compare it on the same plot to the sine function, as well as the quadratic interpolant of $\sin(x)$ with nodes $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi$ (see Worksheet 7 and also add the Hermite interpolant from that worksheet if you computed it).

3. [**Chebyshev orthogonal polynomials, 6pts**] Obtain explicit formulas for the first three Chebyshev polynomials on $[-1, 1]$ via the Gram-Schmidt orthogonalization process starting with the monomials $\{1, x, x^2\}$. Note that unless you normalize these polynomials they are only defined up to a constant, so choose the coefficient in front of the highest power of x to be one.
[Hint: *By examining the integrands and using symmetry you can avoid computing several of the integrals, and only need to compute one integral in the end. The change of coordinates $x = \cos\theta$ may be useful for this integral; if you cannot compute just write the final formulas down and leave the integral unevaluated.*]

4. [**3-point Gaussian quadrature, 8pts**] Recall that Gaussian quadrature is the unique quadrature rule with n points that is exact for polynomials of degree up to and including $2n - 1$.
   (a) [2pts] Show that this statement is equivalent to saying that the quadrature rule is exact for the monomials x^0, x^1, $x^2, \dots, x^{2n-1}$.
   (b) [6pts] Compute the formula for the 3-point Gaussian quadrature rule on the interval $(-1, 1)$. To do this, use symmetry and the fact 3 is an odd number to conclude that the nodes must be the points $\boldsymbol{x} = [-x_1, 0, x_1]$ where $x_1 \in (0, 1)$ is unknown, and the weights must be $\boldsymbol{w} = [w_1, w_2, w_1]$. Write down a system of 3 equations for the three unknowns $\{x_1, w_1, w_2\}$ and solve it. [Hint: *Solving the system does not require any complicated algebra, and the answer is on Wikipedia.*]

5. [**Accuracy of quadrature rules, 8pts**] Write a code to compute the definite integral of a function using the composite Simpson rule with a given number of points n; write your code so you can easily change the function and number of points. Use this code to approximate $\int_0^1 \exp(x)^2 \sin(x)^2 \, dx \approx 1.2668$.
   (a) [5pts] How many points n do you need to get 4 decimal places correctly, i.e., to get the answer 1.267? Give details of how you computed this, and note that there are smarter ways to do this then just trying different values of n. In particular, think about how the error decays as you increase n and try to estimate the right n for an absolute error of $\approx 10^{-4}$.
   (b) [3pts] How many digits do you get for the 3-point Gaussian quadrature rule (see problem 3, lecture notes, or Wikipedia)?