Function Approximation

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Outline

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Function Spaces

- **Function spaces** are the equivalent of finite vector spaces for functions (space of polynomial functions $\mathcal{P}$, space of smoothly twice-differentiable functions $\mathcal{C}^2$, etc.).
- Consider a one-dimensional interval $I = [a, b]$. Standard norms for functions similar to the usual vector norms:
  - **Maximum norm**: $\|f(x)\|_\infty = \max_{x \in I} |f(x)|$
  - L_1 **norm**: $\|f(x)\|_1 = \int_a^b |f(x)|\, dx$
  - **Euclidian** L_2 **norm**: $\|f(x)\|_2 = \left[\int_a^b |f(x)|^2\, dx\right]^{1/2}$
  - **Weighted norm**: $\|f(x)\|_w = \left[\int_a^b |f(x)|^2\, w(x) dx\right]^{1/2}$
- An L_2 **inner or scalar product** (equivalent of dot product for vectors):

$$(f, g)_{L_2} = \int_a^b f(x) g^\star(x) dx$$

Finite-Dimensional Function Spaces

- Formally, function spaces are **infinite-dimensional linear spaces**. Numerically we always **truncate and use a finite basis**.
- Consider a set of $m+1$ **equispaced nodes** $x_i = ih \in \mathcal{X} \subset I$, $i = 0, \ldots, m$, and define:

$$\|f(x)\|_2^{\mathcal{X}} = \left[h \sum_{i=0}^{m} |f(x_i)|^2 \right]^{1/2} = h^{1/2} \|\mathbf{f}_{\mathcal{X}}\|_2 \underset{h \to 0}{\rightarrow} \|f(x)\|_2 ,$$

  which is equivalent to thinking of the function as being the vector $\mathbf{f}_{\mathcal{X}} = \mathbf{y} = \{f(x_0), f(x_1), \cdots, f(x_m)\}$.
- **Finite representations** lead to **semi-norms**, but this is not that important.
- A **discrete dot product** can be just the vector product:

$$(f, g)_{L_2}^{\mathcal{X}} = h\left(\mathbf{f}_{\mathcal{X}} \cdot \mathbf{g}_{\mathcal{X}}\right) = h \sum_{i=0}^{m} f(x_i) g^{\star}(x_i) \underset{h \to 0}{\rightarrow} (f, g)_{L_2}$$

Function Space Basis

- Think of a function as a vector of coefficients in terms of a set of n **basis functions**:

$$\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\},$$

  for example, the monomial basis $\phi_k(x) = x^k$ for polynomials.

- A finite-dimensional approximation to a given function $f(x)$:

$$\tilde{f}(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

- **Least-squares approximation** for $m > n$ (usually $m \gg n$):

$$\mathbf{c}^\star = \arg\min_{\mathbf{c}} \left\| f(x) - \tilde{f}(x) \right\|_2,$$

  which gives the **orthogonal projection** of $f(x)$ onto the finite-dimensional basis.

Choosing the right basis functions

- There are many mathematically equivalent ways to rewrite the unique interpolating polynomial:

$$x^2 - 2x + 4 = (x - 2)^2.$$

- One can think of this as choosing a different **polynomial basis** $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x)\}$ for the function space of polynomials of degree at most m:

$$\phi(x) = \sum_{i=0}^{m} a_i \phi_i(x)$$

- For a given basis, the coefficients **a** can easily be found by solving the linear system

$$\phi(x_j) = \sum_{i=0}^{m} a_i \phi_i(x_j) = y_j \quad \Rightarrow \quad \mathbf{\Phi a} = \mathbf{y}$$

Lagrange basis

- Instead of writing polynomials as sums of monomials, let's consider a more general **polynomial basis** $\{\phi_0(x), \phi_1(x), \ldots, \phi_m(x)\}$:

$$\phi(x) = \sum_{i=0}^{m} a_i \phi_i(x),$$

as in $x^2 - 2x + 4 = (x-2)^2$.

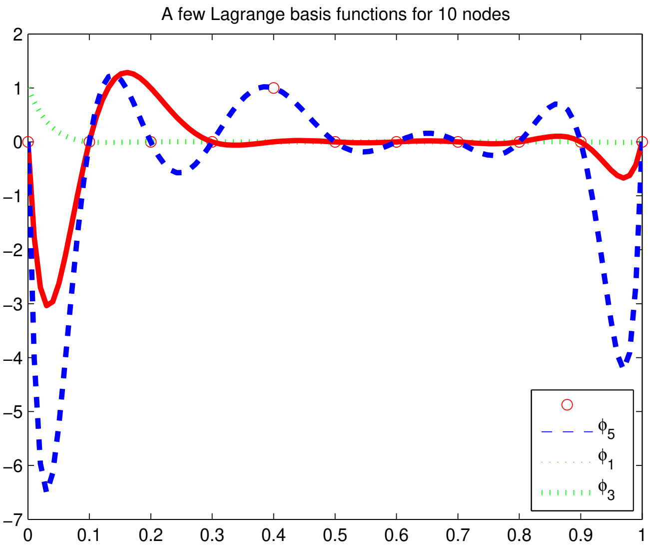
- In particular let's consider the **Lagrange basis** which consists of polynomials that vanish at all but exactly one of the nodes, where they are unity:

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

- The following **characteristic polynomial** provides the desired basis:

$$\phi_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

Lagrange basis on 10 nodes



Outline

Orthogonal Polynomials

- Any finite interval $[a, b]$ can be transformed to $I = [-1, 1]$ by a simple transformation.
- Using a **weight function** $w(x)$, define a **function dot product** as:

$$(f, g) = \int_a^b w(x) \left[f(x) g(x) \right] dx$$

- For different choices of the weight $w(x)$, one can explicitly construct **basis of orthogonal polynomials** where $\phi_k(x)$ is a polynomial of degree k (**triangular basis**):

$$(\phi_i, \phi_j) = \int_a^b w(x) \left[\phi_i(x) \phi_j(x) \right] dx = \delta_{ij} \left\| \phi_i \right\|^2 .$$

- For **Chebyshev polynomials** we set $w = (1 - x^2)^{-1/2}$ and this gives

$$\phi_k(x) = \cos\left(k \arccos x \right).$$

Legendre Polynomials

- For equal weighting $w(x) = 1$, the resulting triangular family of of polynomials are called **Legendre polynomials**:

$$\begin{aligned}
\phi_0(x) =& 1 \\
\phi_1(x) =& x \\
\phi_2(x) =& \frac{1}{2}(3x^2 - 1) \\
\phi_3(x) =& \frac{1}{2}(5x^3 - 3x) \\
\phi_{k+1}(x) =& \frac{2k+1}{k+1} x\phi_k(x) - \frac{k}{k+1}\phi_{k-1}(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]
\end{aligned}$$

- These are orthogonal on $I = [-1, 1]$:

$$\int_{-1}^{-1} \phi_i(x)\phi_j(x)dx = \delta_{ij} \cdot \frac{2}{2i+1}.$$

Interpolation using Orthogonal Polynomials

- Let's look at the **interpolating polynomial** $\phi(x)$ of a function $f(x)$ on a set of $m+1$ **nodes** $\{x_0, \ldots, x_m\} \in I$, expressed in an orthogonal basis:

$$\phi(x) = \sum_{i=0}^{m} a_i \phi_i(x)$$

- Due to orthogonality, taking a dot product with ϕ_j (**weak formulation**):

$$(\phi, \phi_j) = \sum_{i=0}^{m} a_i (\phi_i, \phi_j) = \sum_{i=0}^{m} a_i \delta_{ij} \|\phi_i\|^2 = a_j \|\phi_j\|^2$$

- This is **equivalent to normal equations** if we use the right dot product:

$$(\mathbf{\Phi}^\star \mathbf{\Phi})_{ij} = (\phi_i, \phi_j) = \delta_{ij} \|\phi_i\|^2 \text{ and } \mathbf{\Phi}^\star \mathbf{y} = (\phi, \phi_j)$$

Gauss Integration

$$a_j \left\|\phi_j\right\|^2 = (\phi, \phi_j) \quad \Rightarrow \quad a_j = \left(\left\|\phi_j\right\|^2\right)^{-1} (\phi, \phi_j)$$

- Question: Can we easily compute

$$(\phi, \phi_j) = \int_a^b w(x) \left[\phi(x)\phi_j(x)\right] dx = \int_a^b w(x) p_{2m}(x) dx$$

for a polynomial $p_{2m}(x) = \phi(x)\phi_j(x)$ of degree at most $2m$?

Gauss nodes

- If we choose the **nodes to be zeros of** $\phi_{m+1}(x)$, then we can **quickly project any polynomial** onto the basis of orthogonal polynomials:

$$(\phi, \phi_j) = \sum_{i=0}^{m} w_i \phi(x_i) \phi_j(x_i) = \sum_{i=0}^{m} w_i f(x_i) \phi_j(x_i)$$

  where the **Gauss weights w** are given by

$$w_i = \int_a^b w(x) \phi_i(x) dx.$$

- The orthogonality relation can be expressed as a **sum instead of integral**:

$$(\phi_i, \phi_j) = \sum_{i=0}^{m} w_i \phi_i(x_i) \phi_j(x_i) = \delta_{ij} \left\| \phi_i \right\|^2$$

Gauss-Legendre polynomials

- For any weighting function the polynomial $\phi_k(x)$ has k simple zeros all of which are in $(-1, 1)$, called the (order k) **Gauss nodes**, $\phi_{m+1}(x_i) = 0$.
- The interpolating polynomial $\phi(x_i) = f(x_i)$ on the Gauss nodes is the **Gauss-Legendre interpolant** $\phi_{GL}(x)$.
- We can thus define a new weighted **discrete dot product**

$$\mathbf{f} \cdot \mathbf{g} = \sum_{i=0}^{m} w_i f_i g_i$$

The Gauss-Legendre interpolant is thus easy to compute:

$$\phi_{GL}(x) = \sum_{i=0}^{m} \frac{\mathbf{f} \cdot \boldsymbol{\phi}_i}{\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i} \phi_i(x).$$

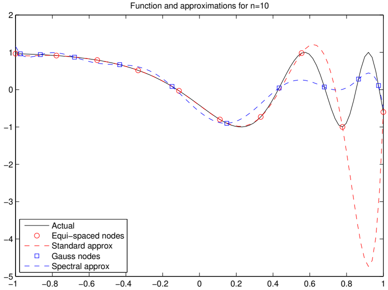
Discrete spectral approximation

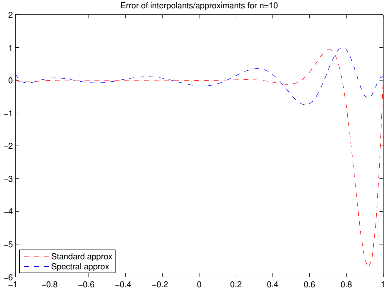
- Using orthogonal polynomails has many advantages for function approximation: **stability**, **rapid convergence**, and **computational efficiency**.
- The convergence, for sufficiently smooth (nice) functions (analytic in the neighborhood of $[-1, 1]$ in the complex plane), is **more rapid than any power law**

$$\|f(x) - \phi_{GL}(x)\| \sim C^{-m},$$

- This so-called **spectral accuracy** (limited by smoothness only) cannot be achived by piecewise, i.e., local, approximations (limited by order of local approximation).

Gauss-Legendre Interpolation





Global polynomial interpolation error