Functional Analysis

Basics

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Function space (space of functions):

- Space of polynomials of degree n: P_n
- Space of all polynomials: P
- Space of (once/twice) continuously differentiable functions $C^{1/2}$
- Space of all continuous functions C^0

$$P_n \subset P$$

$$C^0 \subset C^1 \subset C^2$$

These are <u>vector spaces</u>

$f \in \mathcal{P}^n, \quad \alpha f \in \mathcal{P}^n$

$f_1, f_2 \in \mathcal{P}^n, \quad f_1 + f_2 \in \mathcal{P}^n$

→ Linear Algebra Applies

$\dim\{\mathcal{P}^n\} = n+1$

basis $\{\underbrace{x^0, x^1, x^2, \ldots}_{(n+1) \text{ monomials}}, x^n\}$

$\mathcal{P}^n$ is just like $\mathbb{R}^n$

$P_1, P_2 \in \mathcal{P}^n$

p_1 and p_2 are linearly independent

$$\begin{cases} p_1 = a_0^{(1)} + a_1^{(1)} x + a_2^{(1)} x^2 + \ldots + a_n^{(1)} x^n \\ p_2 = a_0^{(2)} + a_1^{(2)} x + \ldots\ldots + a_n^{(2)} x^n \end{cases}$$

iff the vectors

$\vec{a}^{(1)}$ and $\vec{a}^{(2)} \in \mathbb{R}^n$

are linearly independence.

Inner-products for functions

① $$\|f(x)\|_\infty = \max_{x \in [a,b]} |f(x)|$$

L-infinity or max norm

② $$\|f(x)\|_1 = \int_a^b |f(x)|\, dx$$

③ L_2 norm

$$\|f(x)\|_2 = \sqrt{\int_a^b |f(x)|^2\, dx}$$

④ Weighted L_2 norm:

For vectors in $\mathbb{R}^n$

$$\|\vec{x}\|_2^{\vec{w}} = \sqrt{\sum_{i=1}^{n} w_i |x_i|^2}$$

In matrix notation

$$\left(\|\vec{x}\|_2^{\vec{w}}\right)^2 = \vec{x}^T \overrightarrow{W} \vec{x}$$

where $\overrightarrow{W} = \mathrm{Diag}\{w_i\}$

Can generalize to any positive definite matrix W

$$\|f(x)\|_2^{w} = \left[\int_a^b w(x)|f(x)|^2\,dx\right]^{1/2}$$

where $w(x) > 0$ in $[a, b]$

If $w(x) = \underline{1}$ we get the standard L_2 norm.

$$\|f(x)\|_2^2 = \underbrace{(f, f)}_{\text{inner product}}$$

$$\underset{\substack{L_2 \text{ inner} \\ \text{product}}}{(f, g)_2} = \int_a^b f(x)\, \underset{\substack{\uparrow \\ \text{complex} \\ \text{conjugate}}}{g^*}(x)\, dx$$

The different function norms (L_1, L_2, L_∞) are truly different, unlike in finite-dimensional vector spaces ($\mathbb{R}^n$)

$L_2^{[a,b]}$ function space

$$f(x) \in L_2[a,b]$$

iff $\|f(x)\|_2$ is finite

$$\int_a^b |f(x)|^2 \, dx < \infty$$

Function is square integrable

On a computer, we always represent functions as finite-dimensional vectors.

$$f(x) \longleftrightarrow \{f(x_0), f(x_1), \dots, f(x_n)\}$$

$$= \vec{f} \in \mathbb{R}^{n+1}$$

$$f(x) \approx \vec{f}$$

$$p(x) \approx f(x)$$

polynomial interpolation

$$\|f(x)\|_2^2 \simeq h \|\vec{f}\|_2^2$$

cannot compute exactly on computer

$$= h \sum_{i=0}^{n} |f(x_i)|^2$$

can be evaluated on computer $\approx \int |f(x)|^2 dx$

h is grid spacing

$x_i = a + i \cdot h$

Similarly

$(f, g) \longleftrightarrow h(\vec{f} \cdot \vec{g})$

<u>Theorem</u>:

$$h(\vec{f} \cdot \vec{g}) \xrightarrow[h \to 0]{} (f, g)$$

$$= \int f(x) g^*(x) dx$$