Linear Algebra

(mostly review) A. Donev

Matrix
$$A = \left[\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \vec{a}_3 \,\middle|\, \dots \,\middle|\, \vec{a}_n \right]$$

$\vec{a}_k \in \mathbb{R}^m$, n of them

$$A \in \mathbb{R}^{m,n}$$

Matrix-vector product

$$\vec{b} = A\vec{x} = \underline{x_1}\,\vec{a}_1 + \underline{x_2}\,\vec{a}_2 + \dots + \underline{x_n}\,\vec{a}_n$$

column vector

$$[m \times n]\,[n \times 1] = [m \times 1]$$

$$b \in \mathbb{R}^m$$

Image, column space, or range of A:

$\vec{b} \in \mathrm{im}(A)$

Vector subspace of $\mathbb{R}^m$ spanned by columns of A

Columns of A are linearly independent if

$$Ax = 0 \iff x = 0$$

$$r = \dim(\mathcal{V} \subset \mathbb{R}^n)$$

max number of lin. ind. vectors in $\mathcal{V}$.

# of vector in $\underline{\text{any}}$ basis for V

$\dim(\mathbb{R}^n) = n$. is finite

Inner or dot product

$$\vec{x} \cdot \vec{y} = (x, y) = \langle x, y \rangle$$

$$= x^T y = \sum_{i=1}^{n} x_i y_i$$

$$[1 \times n][n \times 1] = [1 \times 1] = \text{scalar}$$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

If $\mathbb{C}^n$ (complex number)

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} \overline{x_i}\, y_i$$

$\vec{x} \perp \vec{y}$ (orthogonal) if

$\vec{x} \cdot \vec{y} = 0$

$r = \text{rank}(A) =$

$\dim(\text{im}(A))$

# of lin. ind. rows or columns

$\text{rank}(A) = \text{rank}(A^T)$

$r \le \min(m, n)$

If $A\vec{x} = \vec{0} \Rightarrow$

$\vec{x} \in \text{null}(A)$

null space or kernel of A

$\dim(\text{null}(A)) = \underline{\underline{\text{nullity}}}$

If column are lin. ind $\Rightarrow$

$\text{null}(A) = 0$

<u>Fund</u> theorem of LA:

rank + nullity = n

$L : V \longrightarrow W$

vector spaces

L is linear iff

$$\begin{cases} L(\vec{v}_1 + \vec{v}_2) = L\vec{v}_1 + L\vec{v}_2 \\ L(\alpha \vec{v}) = \alpha (L\vec{v}) \end{cases}$$

All linear mappings can be represented by a matrix
(finite-dimensional $\mathcal{V}$ and $\mathcal{W}$)

$$L(\vec{v}) = \underbrace{\overleftrightarrow{L}\,\vec{v}}_{\text{matrix-vector product}}$$

$$\vec{w} = \overleftrightarrow{L}\,\vec{v}$$

$$w_i = \sum_{j=1}^{n} L_{ij} v_j$$

contraction

$$w_i = (L_{i,:}, \vec{v})$$

i^{th} row of matrix

Composition of linear mappings

$$\vec{z} = \underbrace{A(B\vec{x})}_{\text{associative}} = (AB)\vec{x}$$

$$= C\vec{x}$$

$$C = AB$$ matrix-matrix product

$$C_{ij} = \sum_{k=1}^{p} A_{\underline{\underline{i}}\,k} B_{k\,\underline{\underline{j}}}$$

contract

$A = [m \times p]$

$B = [p \times n]$

$C = [m \times p][p \times n] = [m \times n]$

In Matlab $A * B$

Matrix multiplication is not commutative

$$AB \neq BA \quad \text{(in general)}$$

$$A : \mathbb{R}^n \underset{A^{-1}}{\rightleftarrows} \mathbb{R}^n$$

$$A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

$$AA^{-1} = A^{-1}A = I$$

I = identity matrix

$$I = \begin{bmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{bmatrix}$$

If A^{-1} exists, matrix is invertible (square)

<u>A is invertible</u> iff:
(all of these are equivalent)

1) A is full rank
$\text{rank}(A) = n$

2) columns & rows are lin. ind.

3) $\det(A) \neq 0$

4) $\left(\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n\right)$

$\lambda = 0$ is <u>not</u> an eigenvalue

5) $Ax = 0 \iff x = 0$

Properties of matrix algebra

$$C(A+B) = CA + CB$$

$$ABC = (AB)C$$
$$= A(BC)$$

E.x.

$$A\vec{x}\vec{x}^T A = A\underbrace{(\vec{x}\vec{x}^T)}_{B}A$$

$$\underbrace{(\vec{x}^T y)}_{\text{scalar}}B$$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix "division" $\times$

→ multiplication by inverse

$$\overset{(A^{-1})}{A}\ \overset{(B^{-1})}{B} = \overset{(A^{-1})}{C}\ \overset{(B^{-1})}{} $$

$$\cancel{B = C / A}$$

$$(\underbrace{A^{-1} A}) B = A^{-1} C$$

$$\underbrace{I} B = A^{-1} C$$

$$B = A^{-1} C$$

$$A = C B^{-1}$$

Vector norms

p- norm or L_p norm

$p \geq 1$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

1) L_1 norm (Manhattan norm)

$p=1$, $\|\vec{x}\|_1 = \sum |x_i|$

2) L_2 norm (Euclidean norm)

$$\|x\|_2 = \sqrt{x \cdot x} = \sqrt{x^T x}$$

$$\|\vec{x}\|_2 = \sqrt{\sum |x_i|^2}$$

3) L_∞ norm or max norm

$$\|\vec{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

All norms are related to each other if n ($\vec{x} \in \mathbb{C}^n$) is "small"

$\|A\|$ Matrix norm

$$\sup_{x \neq 0} \frac{\|Ax\|_{1,2,\infty}}{\|x\|_{1,2,\infty}} = \|A\|_{1,2,\infty}$$

Matrix norm induced by the vector norm

$$\|Ax\| \le \|A\| \, \|x\|$$

$$\|AB\| \le \|A\| \, \|B\|$$

1) $$\|A\|_1 = \max_j \|A_{:,j}\|_1$$

j^{th} column

$$= \max_j \sum_{i=1}^{n} |a_{ij}|$$

2) $$\|A\|_\infty = \max_i \|A_{i,:}\|_1$$

$$= \max_i \sum_j |a_{ij}|$$

3) $\|A\|_2 = \max_i \lambda_i$

λ^2 is an eigenvalue of

$\underbrace{A^T A}_{\text{symmetric}}$ or $A A^T$

$\|A\|_2 = \max_i \sqrt{\lambda_{A^T A}}$

In Matlab

$$\text{norm}\left(A, \begin{matrix} 1 \\ 2 \\ \text{inf} \end{matrix}\right) \begin{matrix} p=1 \\ p=2 \\ p=\infty \end{matrix}$$

Conditioning of mappings/matrices

$$\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$$

$$\sup_{\delta\vec{x} \neq 0} \frac{\|\vec{f}(\vec{x}+\delta\vec{x}) - f(\vec{x})\|}{\|\delta\vec{x}\|}$$

$$= \underline{\mathrm{Cond}}_x(f)$$

(Local absolute condition number in theory book)

$$f : \mathbb{R} \to \mathbb{R}$$

$$\underline{\mathrm{Cond}}(f) = |f'(x)|$$

$$\operatorname{cond}_x(f) = \sup_{\delta x \neq 0} \frac{\|f(x+\delta x) - f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|}$$

If $\operatorname{cond}_x(f) = 10^4$

that means that

knowing 16 digits in x

gives me $16 - 4 = 12$ digits

in $f(x)$

Loose four digits

$$\cancel{A}x + A\delta x - \cancel{A}x$$

$$\operatorname{cond}_x(Ax) = \sup_{\delta x \neq 0} \frac{\|A(x+\delta x) - Ax\| / \|Ax\|}{\|\delta x\| / \|x\|}$$

$$= \sup_{\delta x \neq 0} \left(\frac{\|A \delta x\|}{\|\delta x\|} \right) \frac{\|x\|}{\|Ax\|}$$

$$= \frac{\|x\|}{\|Ax\|} \qquad \|A\| \geq 1$$

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\operatorname{cond}_x(Ax) = \frac{\|A^{-1}(Ax)\| \, \|A\|}{\|Ax\|}$$

$$\leq \frac{\|A^{-1}\| \, \|\cancel{Ax}\| \, \|A\|}{\|\cancel{Ax}\|}$$

$$\leq \|A\| \|A^{-1}\|$$

$$1 \leq \text{cond}_x(A) \leq \underline{\underline{\|A\|\|A^{-1}\|}}$$

Define cond. number of A

$$\underset{1,2,\infty}{K(A)} = \underset{1,2,\infty}{\|A\|} \|A^{-1}\|_{1,2,\infty}$$