Nonlinear equations

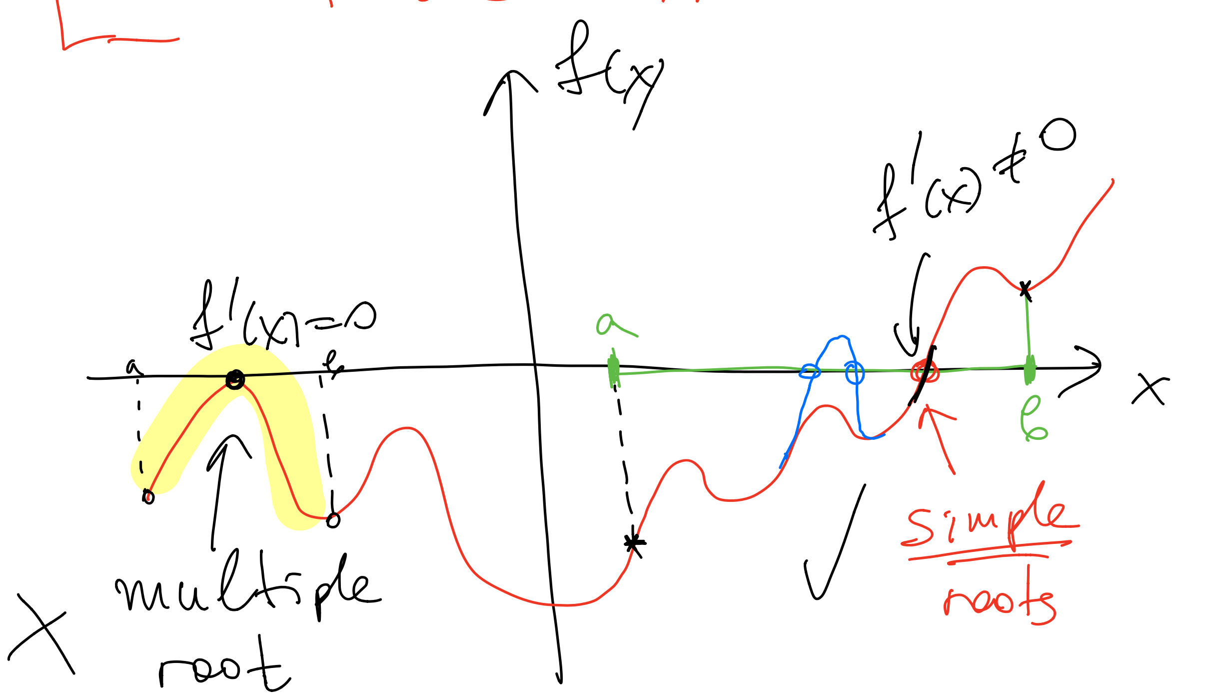
(in one variable)

$x = \sqrt{c} \iff x^2 - c = 0$

$\cos x + x^2 - 7 = 0$

Solve $f(x) = 0 \quad x \in [a, b]$

Find at least one solution if one exists



$f(x)$ is continuous on $[a,b]$

If $f(a) \cdot f(b) < 0$

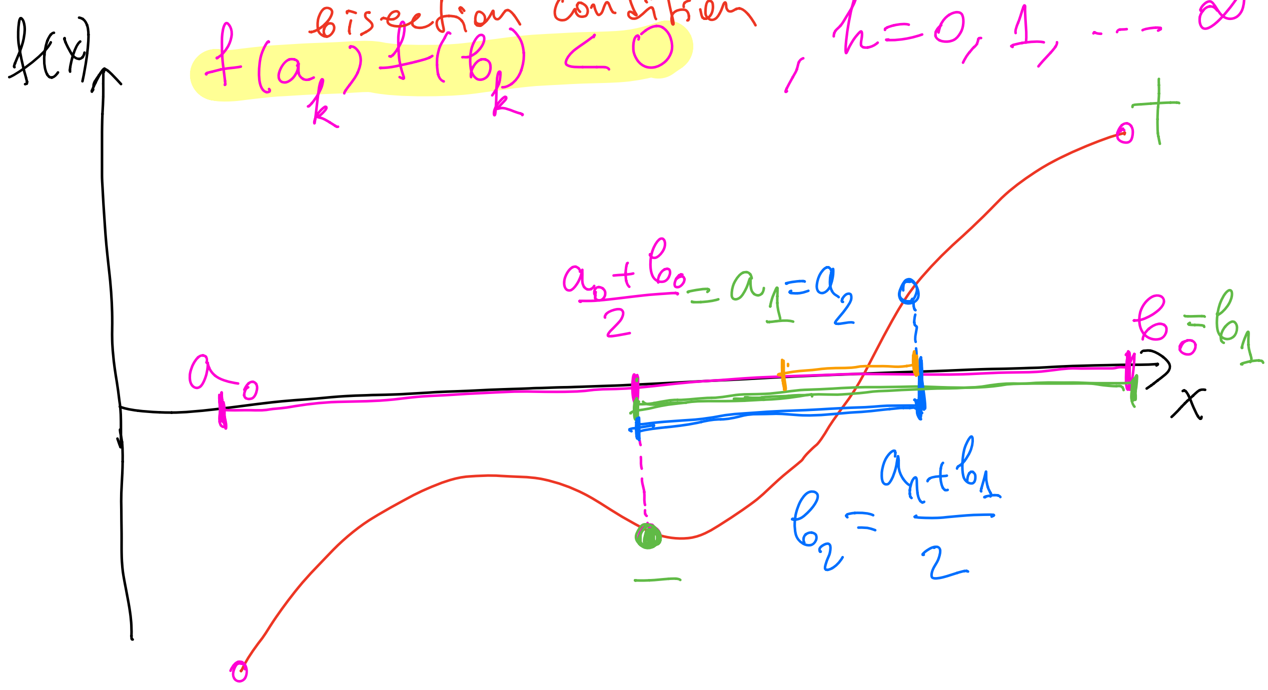
$\Rightarrow \exists x \in [a,b]$ s.t. $f(x) = 0$

(Q) Why don't I write $f(a) f(b) \leq 0$?

Bisection method

Bisection condition

$f(a_k) f(b_k) < 0$, $k = 0, 1, \ldots \infty$



Algorithm

Input: $\begin{cases} [a, b] \text{ s.t. } f(a)f(b) < 0 \\ k_{max} \in \mathbb{Z}^+ \end{cases}$

Output: $\begin{cases} \tilde{x} \text{ s.t. } f(\tilde{x}) \simeq 0 \\ [a, b] \text{ s.t. true root is in } [a, b] \\ x \in [a, b],\ f(x) = 0 \end{cases}$

For $k = 0, 1, 2, \dots, k_{max}$

$$x_k = \frac{a_k + b_k}{2}$$

$$f_k = f(x_k)$$

(actually, we only need sign of f_k)

If $\quad f_k \cdot f(a_k) < 0$ then

left half $[a_k, x_k]$ is a bisection interval,
$x \in [a_k, x_k]$

$$a_{k+1} = a_k \quad ; \quad b_{k+1} = x_k$$

else [if $f_k \cdot f(b_k) < 0$] (keep right half)

$$b_{k+1} = b_k \quad ; \quad a_{k+1} = x_k$$

end if

[else if $f_k = 0$ then
return x_k

End for

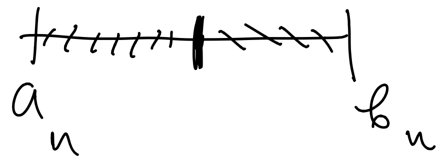
Output : $x \approx x_{k_{max}} = \dfrac{a_{k_{max}} + b_{k_{max}}}{2}$

and $\quad x \in [a_{k_{max}}, b_{k_{max}}]$

We know that [$n = k_{max}$]

$$|x - x_n| \le \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2 \cdot 2^n}$$

$$x_n = \frac{a_n + b_n}{2}$$



Absolute error

$$e_n = |x - x_n| \le \frac{b - a}{2^{n+1}}$$

error estimate

Given an error tolerance ε s.t. $|x - x_n| < \varepsilon$

$$\varepsilon = \frac{b-a}{2^{n+1}}$$

$$2^{n+1} > \frac{b-a}{\varepsilon}$$

$$n+1 = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

How many times I will evaluate f(x)

$$n = \left\lceil \log_2\left(\frac{b-a}{\varepsilon}\right) \right\rceil$$

round up $\lceil \; \rceil$ / ceil

in Matlab

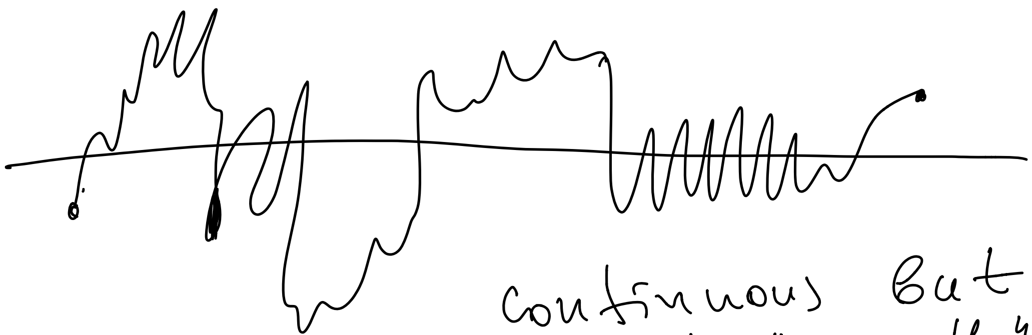
round down $\lfloor \; \rfloor$ / floor

In Matlab :

n_{est} = ceil(log2((b-a)/ε))

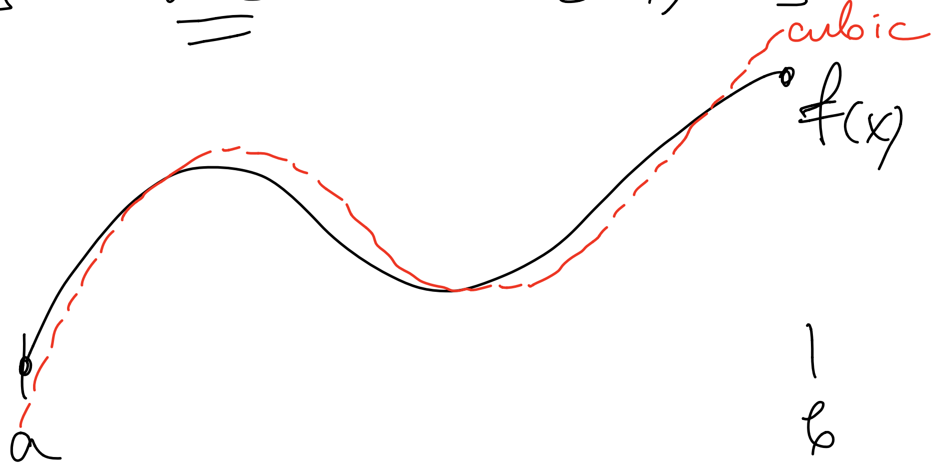
Our job is done

Can we do better?



continuous but not "smooth"

f(x) is "smooth" on [a, b] if it can be approximated "well" by a polynomial of low degree (linear, quadratic, cubic) over <u>all</u> of [a, b]



Another (related) is that $f(x)$ is smooth if it is sufficiently differentiable

$e^{1/x}$ is a bad example

<u>Idea:</u> $p(x) \simeq f(x)$ on $[a, b)$

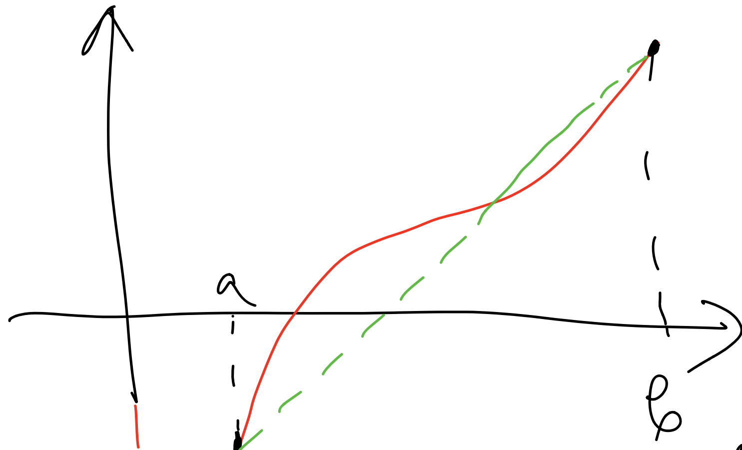
polynomial

Solve $p(x) = 0$ instead

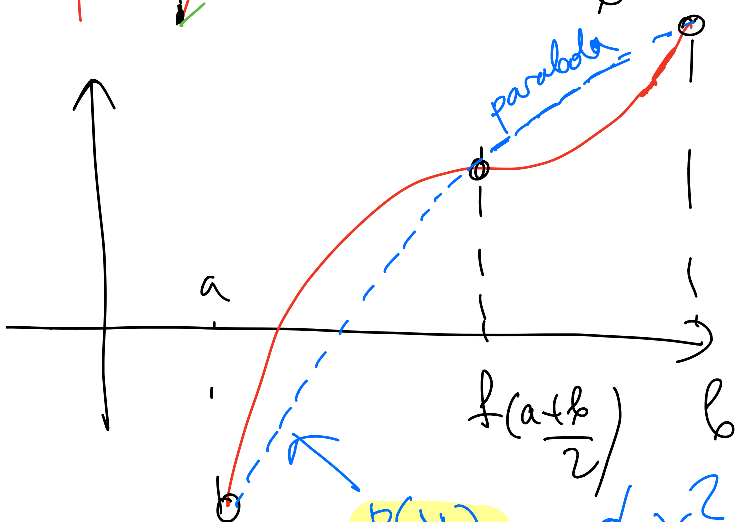
Typically $p(x)$ is linear or quadratic

How do we find $p(x)$?

Option 1

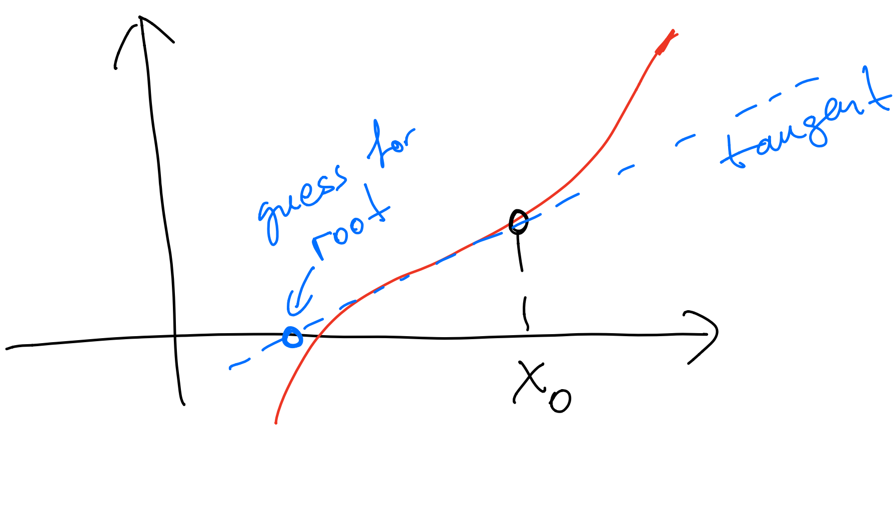


Interpolation



$P(x) = \alpha x^2 + \beta x + \gamma$

Option 2 :



Option 2 : Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$p(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \ldots$$

$$p(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

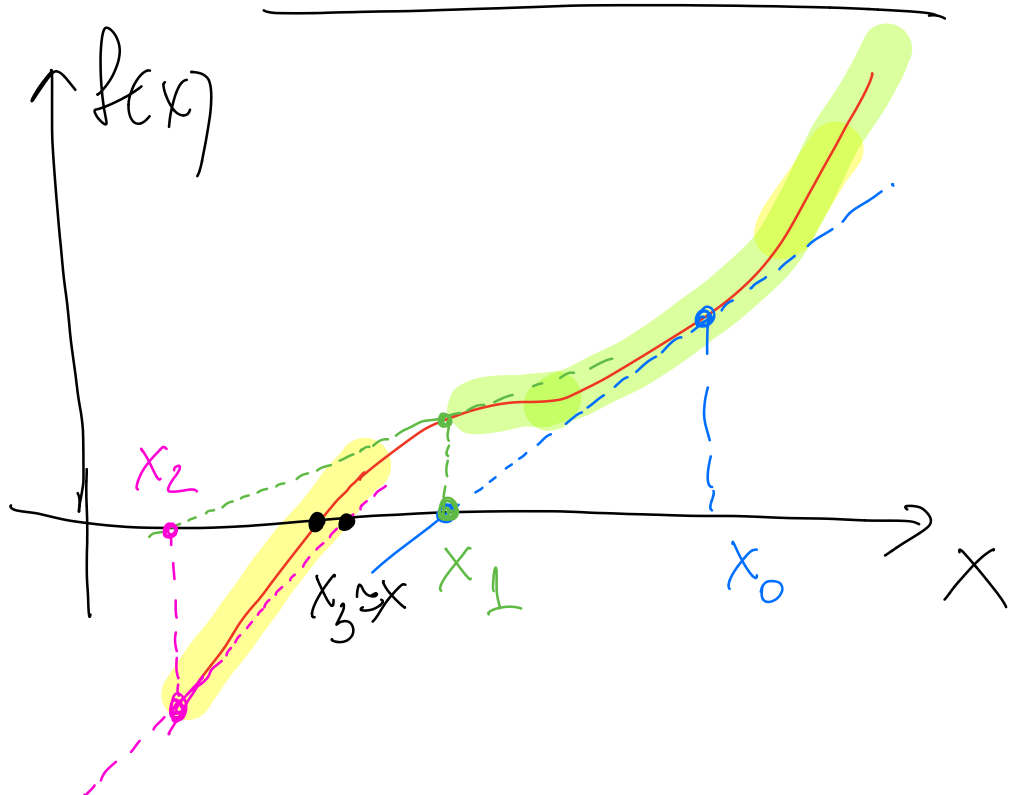
Truncated Taylor series

$$f(x) - p(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$$

Remander

ξ is between x and x_0

Newton's method



Given x_k, $k = 0, 1, 2, \ldots$

Compute x_{k+1}

$$f(x) \simeq p_k(x) = f(x_k) + f'(x_k)(x - x_k)$$

$p_k(x) = 0$ solve for x

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Newton's method

E.g. $f(x) = x^2 - c$

$$f' = 2x$$

$$x_{k+1} = x_k - \frac{(x_k^2 - c)}{2x_k}$$

$$= \frac{1}{2}\left(x_k + \frac{c}{x_k}\right)$$

Babylonian $\Longleftrightarrow$ Newton's

Numerical Analysis : Include more terms in Taylor series

$$e_k = x_k - x$$

(sometimes $e_k = |x - x_k|$)

$$f(x) = 0 = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}(x - x_k)^2 f''(\xi)$$

ξ is between x and x_k

$$x_k = e_k - x$$

Divide by $f'(x_k)$

$$\underbrace{\left(x_k - \frac{f(x_k)}{f'(x_k)}\right)} - x = \frac{1}{2}\underbrace{(x - x_k)}_{e_k}{}^2 \frac{f''(\xi)}{f'(x_k)}$$

$$x_{k+1} - x = e_{k+1}$$

$$e_{k+1} = -\frac{1}{2}\frac{f''(\xi)}{f'(x_k)} e_k^2$$

$$\frac{e_{k+1}}{e_k^2} = -\frac{f''(\xi)}{2f'(x_k)} \to C$$

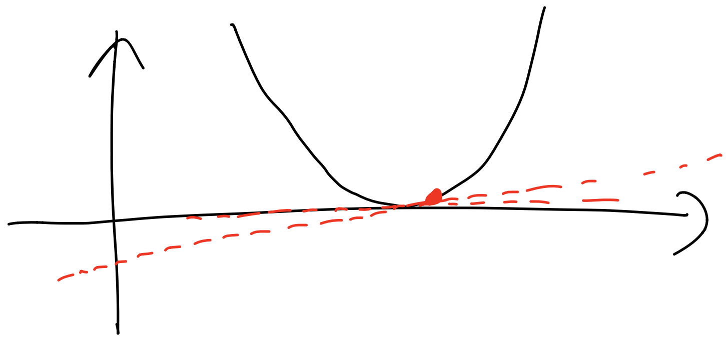
Assume method converges

$$x_k \to x$$

$$\xi \to x$$

$$\frac{|e_{k+1}|}{|e_k^2|} \to \frac{1}{2}\frac{f''(x)}{f'(x)} = C$$

Double root: $f'(x) = 0$



Won't converge

$$|e_{k+1}| < |e_k|$$

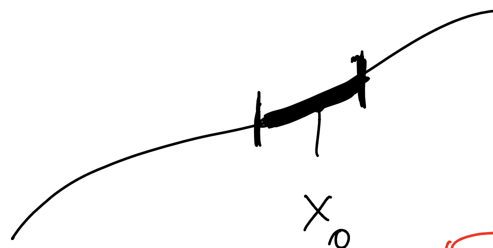
We want $|e_{k+1}| \leq \dfrac{|e_k|}{2}$

$$\left|\frac{f''(\xi)}{2f'(x_k)}\right| e_k^{\cancel{2}} < \frac{\cancel{|e_k|}}{2}$$

$$|e_k| < \left|\frac{2f'(x_k)}{2f''(\xi)}\right| < A$$

$$e_0 = x_0 - x$$

$$|x_0 - x| < \left| \frac{2 f'(x_0)}{2 f''(\xi)} \right|$$



Theorem 1.8

Estimate: Practice

$$\boxed{|x_0 - x| < \left| \frac{f'(x)}{f''(x)} \right|}$$

A

Assumption in Suli:

$$\left| \frac{f''(y_1)}{f'(y_2)} \right| < A$$

$$\forall y_1, y_2 \in [x - \delta, x + \delta]$$

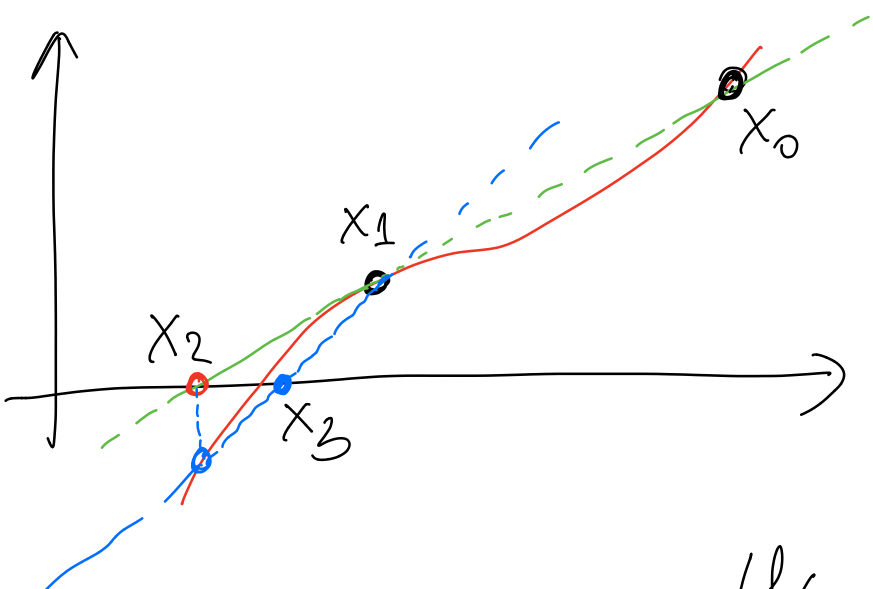
Theorem: If

$$|e_0| = |x_0 - x| < \frac{1}{A}$$

then Newton's method will converge

Newton's method needs to be started close to root and then it will converge fast

Secant method



$$f(x) = p(x) = f(x_k) + \frac{(f(x_{k+1}) - f(x_k))}{(x_{k+1} - x_k) \to 0}(x - x_k)$$

If $x = x_k \Rightarrow p(x_k) = f(x_k)$

If $x = x_{k+1} \Rightarrow p(x_{k+1}) = f(x_{k+1})$

Newton $p(x) = f(x_k) + f'(x_k)(x - x_k)$

Conclusion: As $x_k \to x$

$x_{k+1} \to x$

These two become closer

$p(x_{k+2}) = 0$ solve for x_{k+2}

secant method

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{(f_k - f_{k-1})} f_k$$

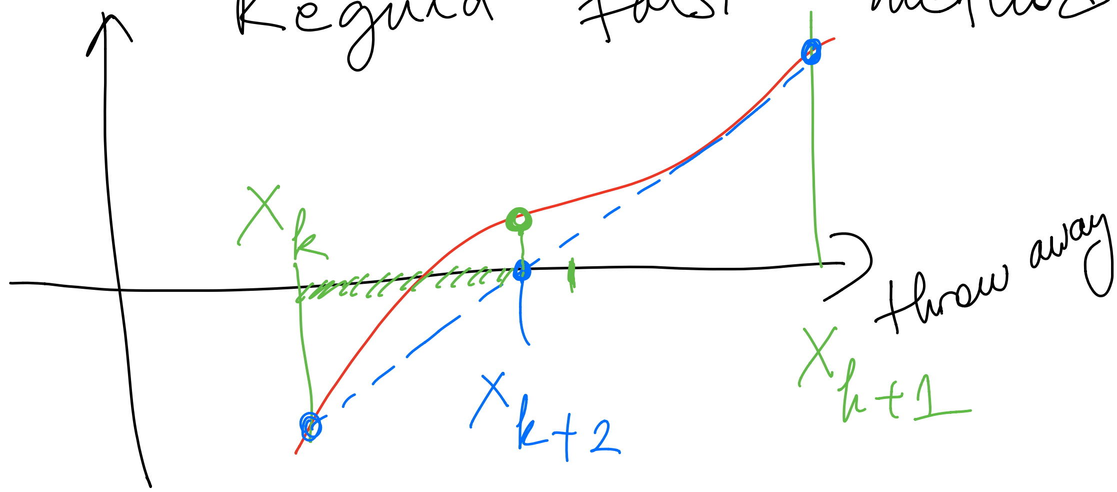
converges to $\frac{f(x)}{f'(x)}$

$$f_k = f(x_k)$$

We can combine bisection + secant

or "Safeguarded" secant

"Regula falsi" method



Problem 3 in HW 1:

Secant method:

$$|e_{k+1}| \longrightarrow \mu |e_k|^q$$

Bisection: $q = 1$, $\mu = 1/2$

Steffensen
Newton: $q = 2$, $\mu = \frac{f''(x)}{2f'(x)}$

(quadratically convergent)

Secant: $q = \frac{1}{2}(1+\sqrt{5}) \simeq 1.6$

Fixed-point iteration

$$x_{k+1} = g(x_k)$$

$$x_k \to x \quad , \quad f(x) = 0$$

$$x = g(x)$$

equivalent

Babylonian

$$g(x) = \frac{1}{2}\left(x + \frac{c}{x}\right)$$

$$x = g(x) \Leftrightarrow x = \sqrt{c}$$

$$\underline{x} = x^2 - c + \underline{x} = g(x)$$

$$0 = x^2 - c = f(x)$$

$$x_{k+1} = g(x_k)$$

Fixed-point iteration

Example: Relaxation method

$$x_{k+1} = g(x_k) = x_k - \lambda f(x_k)$$

$$g(x) = x - \lambda f(x)$$

$\lambda \neq 0$

$$x = g(x) \Rightarrow$$

$$\cancel{x} = \cancel{x} - \lambda f(x) = 0$$

$$\Rightarrow f(x) = 0$$

Method is consistent

What is a good λ

$$x_{k+1} = x_k - \lambda f(x_k)$$

$$x_{k+1} = g(x_k) \leftarrow \text{Taylor series around } \underline{\underline{x}} \text{ (solution)}$$

$$= g(x) + (x_k - x)\, g'(\xi)$$

ξ is in-between x, x_k

Express in terms of error

$$\begin{cases} x_{k+1} = \underline{x} + \underbrace{e_{k+1}}_{\text{error}} \\ x_k = x + e_k \end{cases}$$

$$e_{k+1} = x_{k+1} - x =$$

$$= g(x) + (x_k - x)\, g'(\xi)$$

$$- x$$

[we know $x = g(x)$] $\Leftrightarrow f(x) = 0$

$$= (x_k - x)\, g'(\xi)$$

$$= e_k\, g'(\xi)$$

$$|e_{k+1}| = |g'(\xi)|\,|e_k|$$

We want

$$|e_{k+1}| < |e_k|$$

$$\Rightarrow |g'(\xi)| < 1 \quad (*)$$

ξ is between x and x_k

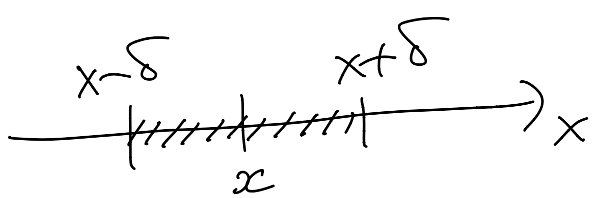
(*) should be true if ξ is close to root

Theorem:

If $g \in C^1$ ← continuously differentiable

$|g'(y)| < 1$ $\forall y \in [x-\delta, x+\delta]$

where $x = g(x)$

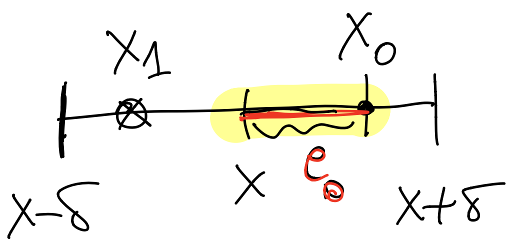


then fixed-point iteration

$$x_{k+1} = g(x_k)$$

... will converge to x if

"Proof": $x_0 \in [x-\delta, x+\delta]$



$$|e_0| < \delta$$

$$|e_1| = |g'(\xi_0)|\,|e_0|$$

$$\wedge$$

$$1$$

$$|e_1| < |e_0| < \delta$$

$$x_1 \in [x-\delta, x+\delta]$$

$$\Rightarrow \quad |e_2| = |g'(\xi_1)|\,|e_1|$$

$$\wedge$$

$$1$$

$$|e_2| < |e_1| < |e_0|$$

By induction

$$|e_{k+1}| \leq L\,|e_k|$$

$$0 < L < 1$$

$$L = \max_{y \in [x-\delta, x+\delta]} |g'(y)| < 1$$

$$|e_k| \leq \underline{\underline{L}}^k\,|e_0|$$

$$L^k \to 0$$

$$\Rightarrow |e_k| \to 0$$ Q.E.D.

$$\frac{|e_{k+1}|}{|e_k|} \rightarrow |g'(x)|$$

since $x_k \to x$

$\xi \to x$

Fast convergence we want

$$|g'(x)| \ll 1$$

Ideally $g'(x) = 0$

convergence is faster than linear $\Rightarrow$ superlinear convergence.

Example: Relaxation

$$x_{k+1} = x_k - \lambda f(x_k)$$

$$g(x) = x - \lambda f(x)$$

$$g'(x) = 1 - \lambda f'(x) = 0$$

$$\lambda = \frac{1}{f'(x)}$$

Best λ. Fastest (super linear) convergence for

Doesn't work

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x)}$$

$\Leftarrow$ we don't know x

c.f. Newton

$$x_k - \frac{f(x_k)}{f'(x_k)}$$

For Newton's method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

what is $g'(x)$ at the root x that satisfies $x = g(x)$

$$g'(x) = \cancel{1} - \cancel{\frac{f'(x)}{f'(x)}} + \frac{f(x)\, f''(x)}{\left(f'(x)\right)^2}$$

$$= \frac{f(x)\, f''(x)}{\left(f'(x)\right)^2}$$

If $f(x) = 0 \Rightarrow g'(x) = 0$

Consistent with Newton's method converging quadratically.