Solving systems of <u>non</u>-linear eqs.

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$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \qquad \begin{cases} \underline{\underline{f}}(x,y) = 0 \\ \underline{\underline{g}}(\underline{\underline{x}}, \underline{\underline{y}}) = 0 \end{cases}$$

$$\begin{matrix} x_1^{(k)} \\ x_2^{(k)} \end{matrix} \longrightarrow \begin{matrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{matrix}$$

No bisection in higher dims.

(real numbers are ordered)

$$f_1(x_1, x_2) = f\left(x_1^{(k)}, x_2^{(k)}\right)$$

$$+ \frac{\partial f_1}{\partial x_1}\left(x_1 - x_1^{(k)}\right)$$

$$+ \frac{\partial f_1}{\partial x_2}\left(x_2 - x_2^{(k)}\right) + O\left(\left(x_1 - x_1^{(k)}\right)^2\right)$$

$$+ O(\text{squared})$$

$$\vec{f} = [f_1, f_2]^T$$

$$\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\vec{x} = (x_1, x_2)^T$$

$$\vec{f}(\vec{x}) : \vec{x} \in \mathbb{R}^2 \to \vec{f} \in \mathbb{R}^2$$

$$\vec{f}(\vec{x}^{(k+1)}) = \vec{f}(\vec{x}^{(k)})$$

$$+ \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} - x_1^{(k)} \\ x_2^{(k+1)} - x_2^{(k)} \end{bmatrix}$$

$$\overleftrightarrow{J}(\vec{x}) \text{ — Jacobian matrix}$$

$$J_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$\vec{f}(\vec{x}^{(k+1)}) = \vec{f}(\vec{x}^{(k)}) + \overleftrightarrow{J}(\vec{x}^{(k)}) \cdot (\vec{x}^{(k+1)} - \vec{x}^{(k)})$$

$$\Delta \vec{X}^{(k)} = \vec{X}^{(k+1)} - \vec{X}^{(k)}$$

$$\vec{f}\left(\vec{X}^{(k+1)}\right) = \vec{0}$$

$$f^{(k+1)} = f^{(k)} + J^{k} \Delta X^{k} = 0$$

$$J^{k} \Delta X^{k} = - f^{(k)}$$

linear system (LU)

$$X^{k+1} = X^{k} + \underline{\Delta X^{k}}$$

$$X^{k+1} = X^{k} - \cancel{\left(J^{k}\right)^{-1}} f^{(k)}$$

on paper

(use backslash / LU)

$$\|A x\| \leq \|A\| \, \|x\|$$

When to stop?

Termination or stopping criterion

Two options:

① Stop if $\| \vec{f}^{\,k} \|_{1,2,\infty} < \varepsilon$

tolerance

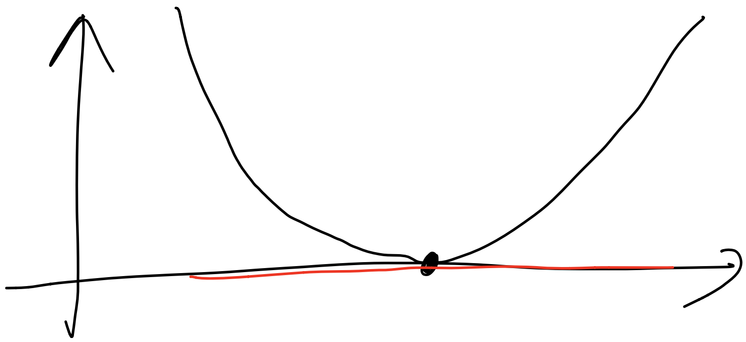
② Stop if $\| \Delta x^k \|_{1,2,\infty} < \varepsilon$

Options are related but not the same

Option 2: $\| (\underbrace{J^k}_{})^{-1} f^k \| < \varepsilon$

estimate $\| (J^k)^{-1} \| \, \| f^k \| < \varepsilon$

$\| f^k \| < \varepsilon / \| (J^k)^{-1} \|$



Second-order Taylor series

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$f(\vec{x} + \Delta\vec{x}) = f(\vec{x}) + \underbrace{\vec{g}^{T} \Delta\vec{x}}_{\text{scalar}} + \frac{1}{2} \underbrace{\underbrace{(\Delta\vec{x})^{T}}_{\text{vector}} \cdot \underbrace{\overleftrightarrow{H} (\Delta\vec{x})}_{\text{vector}}}_{\text{scalar}}$$

$\vec{g} = \vec{J}^{T} =$ gradient (column vector)

$$+ O(\|\Delta\vec{x}\|^3)$$

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = H_{ji}$$

Hessian matrix

symmetric

(Cholesky factorization instead of LU)

Convergence of Newton's method

$$x^{(k+1)} = x^{(k)} - \left(J^{(k)}\right)^{-1} f^{(k)}$$

$$e^{(k)} = x^{(k)} - x \, , \quad f(x) = 0$$

$$\| e^{(k+1)} \| \xleftrightarrow{?} \| e^{(k)} \|$$

$$\vec{f}(\vec{x}^k) = f(x) + J\, e^k$$

$$+ \underbrace{\frac{1}{2} (\vec{e}^k)^T \overleftrightarrow{H} (\vec{e}^k)}_{\cancel{\text{scalar}}\ \text{vector}}$$

$$+ O(\|e^k\|^3)$$

rank-3 tensor
(supported in Matlab)

$$J = J(x)$$

$$H = H(x)$$

$$x^{k+1} - x = e^{k+1} =$$

$$(x^k - x) - J^{-1} f(x^k)$$

$$= e^k - J^{-1} \underbrace{f(x^k)}_{\text{plug in Taylor series}}$$

$$= \underline{\underline{e^k}} - J^{-1}\left[\cancel{f(x)}^{\,0} + \underline{J e^k} + \frac{1}{2}(e^k)^T H (e^k) + O(\|e^k\|^3)\right]$$

$$e^{k+1} = -\frac{J^{-1}}{2}\underbrace{(e^k)^T H (e^k)}$$

$$\|e^{k+1}\| \le \frac{1}{2}\|e^k\|^2 \|H\| \|J^{-1}\|$$

$$\|e^{k+1}\| \le \frac{1}{2}\underline{\|H\|}\,\underline{\|J^{-1}\|}\,\|e^k\|^2$$

quadratic convergence

$$e^{k+1} \approx -\frac{1}{2}\frac{f''(x)}{f'(x)}\left(e^{k}\right)^2$$

If Newton's method converges, it will converge fast (quadratically)

Requires a good initial guess

Efficient but <u>NOT</u> robust