Orthogonal Polynomials

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Interpolation works for a "black-box" function.
Equi-spaced nodes not a good choice in general

$$p_n(x_k) = f(x_k)$$

nodes

? $\varepsilon(x) = |p_n(x) - f(x)|$

$x \neq x_k$

Piece-wise polynomial interp works for any nodes, but $\varepsilon(x)$ is not very small

Another way to approximate

$f(x) \simeq p_n(x) \in \mathcal{P}_n$ on $[a,b]$

$$p_n^* = \arg \min_{p_n \in \mathcal{P}_n} \| f(x) - p_n(x) \|$$

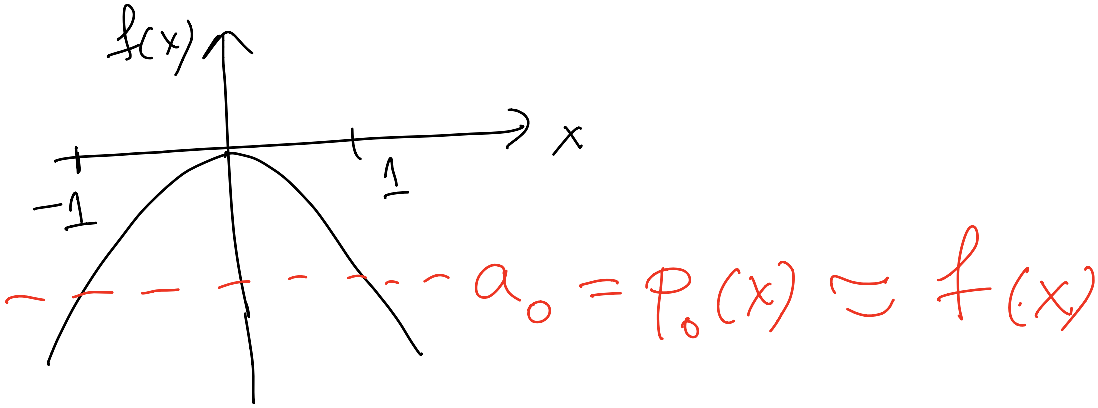
L_2 - approximation

$$p_n^* = \arg \min_{p_n \in \mathcal{P}_n} \| f(x) - p_n(x) \|_2$$

$$\| f(x) - p_n(x) \|_2 = \left(\int | f(x) - p_n(x) |^2 dx \right)^{1/2}$$

Least-squares "fitting"

E.g. Approximate $f(x) = -2x^2$ in $P_{n=0}$ (space of constants) on $[-1, 1]$



Idea: Average of function

$$a_0 = \frac{\int_{-1}^{1} f(x)\,dx}{2} \quad (?)$$

$$= \frac{\int_{-1}^{1} -(\cancel{2}x)^2\,dx}{\cancel{2}} = -\frac{x^3}{3}\Big|_{-1}^{1} = -\frac{2}{3}$$

$\underline{L_2}$:

$$\|f - a_0\|^2 = \int_{-1}^{1} \left(-2x^2 - a_0\right)^2 dx$$

$$a_0^* = \arg\min_{a_0} \; 2a_0^2 + \frac{8a_0}{3} + \frac{8}{5}$$

Derivative $= 0 \Rightarrow a_0^* = -\frac{2}{3}$

$$p_0^* \simeq a_0^* = -\frac{2}{3}$$

More generally:

$$a_0^* = \arg\min \int \left(f(x) - a_0\right)^2 dx$$

$$= \arg\min_{a_0} \left[\int f^2(x)\,dx - 2a_0 \int f(x)\,dx + a_0^2 \int dx \right]$$

$$\frac{\partial}{\partial a_0}\left[\cancel{\int f^2 dx} - 2a_0\int f\,dx + a_0^2\int dx\right]$$

$$-\cancel{2}\int f\,dx + \cancel{2}a_0\int dx = 0$$

$$\Rightarrow a_0 = \frac{\int f\,dx}{\int dx} = \frac{\int_a^b f(x)dx}{b-a}$$

average of function

Now make even more general

$p_n^* \in \mathcal{P}_n$

<u>Step 1</u>:

$$\text{basis} = \{\overbrace{p_0(x), p_1(x), \ldots, p_n(x)}^{n+1}\} \subset \mathcal{P}_n$$

and linearly independent

$$P_n^*(x) = \sum_{j=0}^{n} a_j P_j(x)$$ Best L_2 approx

$$F(\vec{a}) = \int_a^b \Big(f(x) - \sum_{j=0}^{n} \underbrace{a_j}_{\text{unknown}} \overbrace{P_j(x)}^{\text{given}} \Big)^2 dx$$

"normal equations"

$$\frac{\partial F}{\partial a_k} = 0, \quad k = 0, \dots, n$$

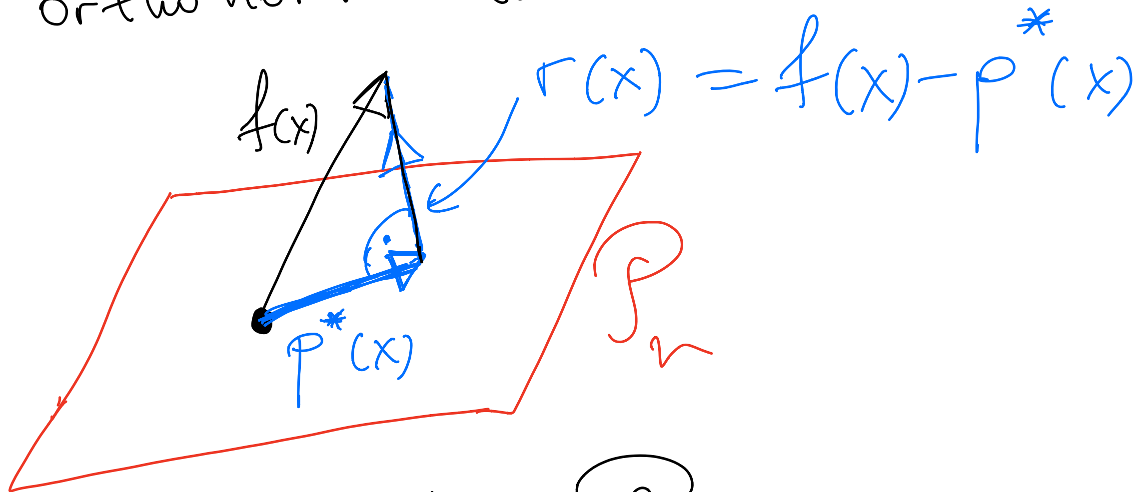
Linear System of $(n+1)$ equation for $(n+1)$ unknowns $\vec{a}$

Reminder: $Ax = b$, overdetermined

$$x^* = \arg\min_x \|Ax - b\|^2$$

① Normal eqs: $(A^T A)x = A^T b$

② QR factorization, Gram-Schmidt orthonormalization.



$$r(x) \perp \mathcal{P}_n$$

Conclusion: $r(x)$ must be orthogonal to all polynomials of degree n $\Longleftrightarrow$

$$r(x) \perp P_k(x), \quad k = 0, \dots, n$$

$$\left(r(x), P_k(x) \right) = 0 \quad \forall k$$

$$\left(f - p^*, p_k\right)_2 = 0 \quad \forall k$$

$$\left(f - \sum_{j=0}^{n} a_j p_j, \; p_k\right) = 0$$

$$(f, p_k) = \sum_{j=0}^{n} a_j (p_j, p_k)$$

$$\sum_{j=0}^{n} (p_j, p_k)\, a_j = (f, p_k)$$

$$k = 0, \dots, n$$

$(n+1)$ equations for $(n+1)$ coeffs.

$$\overleftrightarrow{V}\,\vec{a} = \vec{f}$$

Vandermonde matrix

$$V_{jk} = (P_j, P_k) = \int_{x=a}^{b} P_j(x)\, P_k(x)\, dx$$

Can pre-compute exactly

$$f_k = (f, P_k) = \int_{x=a}^{b} f(x)\, P_k(x)\, dx$$

cannot compute exactly for any $f(x)$

E.g.

Take monomials as basis for

P_n : $\{1, x, x^2, \ldots, x^n\}$

$\{P_0(x), P_1(x), \ldots, P_n(x)\}$

$a=0$, $b=1$

$$V_{ij} = \int_0^1 P_i(x)\,P_j(x)\,dx =$$

$$= \int_0^1 x^i\, x^j\, dx = \int_0^1 x^{i+j}\, dx$$

$$= \frac{x^{i+j+1}}{i+j+1}\Bigg|_0^1 = \frac{1}{i+j+1}$$

$$V_{ij} = \frac{1}{i+j+1} \leftarrow \text{Hilbert matrix}$$

Very ill conditioned for $n \geq 10$
(Worksheets)

Using monomials as a basis is a bad idea (ill-conditioned)

$$\overleftrightarrow{V}\,\vec{a} = \vec{f}$$

What's the simplest **V** ?

$$V = I \Rightarrow \vec{a} = \vec{f}$$

$$V_{ij} = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\begin{cases} \int p_i(x)\, p_j(x)\, dx = 0 & \text{if } i \neq j \\ \int p_i^2(x)\, dx = 1 & \end{cases}$$

← skip this in practice

orthonormal polynomials

$$a_k = f_k = (f, p_k) = \int f(x)\, p_k(x)\, dx$$

$$P_n^* = \sum_{k=0}^{n} a_k\, p_k(x)$$

aside: Often polynomials are orthogonal but not normalized

$$a_k = \frac{(f, p_k)}{(p_k, p_k)}$$

Orthogonal polynomials

How to find an orthogonal basis for $\mathcal{P}_n$?

Recall:

How to find an orthogonal basis for $\operatorname{col} \vec{A}$?

$$A = QR$$

upper triangular

orthonormal

Compute by Gram-Schmidt process

"columns of A" $\stackrel{?}{\equiv} \{1, x, x^2, \ldots, x^n\}$

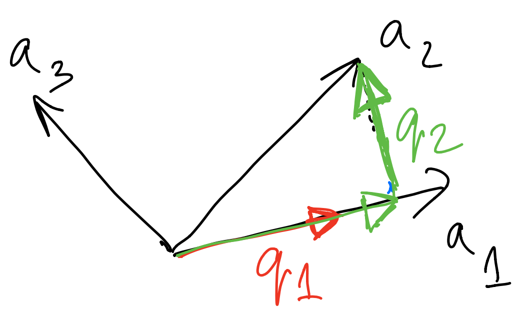
$\text{col}(A) \equiv P_n$

$\downarrow$ GS

$Q \equiv$ orthonormal basis for P_n

Start with monomials, then do GS on them one by one to get orthogonal polynomials

Review:



q_1, q_2, q_3

$q_1 \perp q_2 \perp q_3$

$q_1 = a_1$

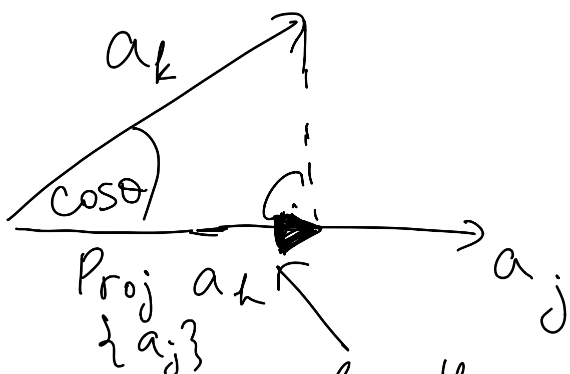
skip normalization

$$q_2 = a_2 - \text{Proj}_{\{a_1\}} a_2$$

$$q_3 = a_3 - \text{Proj}_{\{a_1, a_2\}} a_3 =$$

$$= a_3 - \mathrm{Proj}_{\{a_1\}} a_3 - \mathrm{Proj}_{\{a_2\}} a_3$$

$$\rightarrow \mathrm{Proj}_{\{\vec{a}_j\}} \vec{a}_k = \underbrace{\alpha_{kj}}_{\text{coefficient}} \vec{a}_j$$



$$\text{length} = \|a_k\| \cdot \cos\theta$$

$$(\alpha_{kj} a_j, a_j) = (a_k, a_j)$$

$$\alpha_{kj} = \frac{a_k \cdot a_j}{a_j \cdot a_j} = \frac{(a_k, a_j)}{\|a_j\|^2}$$

$$\mathrm{Proj}_{\{\vec{a}_j\}} \vec{a}_k = \frac{(\vec{a}_k, \vec{a}_j)}{\|a_j\|^2} \vec{a}_j$$

GS for polynomials

$$\{1, x, x^2, x^3, \ldots\}$$

$$\{P_0, P_1, P_2, P_3, \ldots\}$$

Orthogonal polynomials

$$\{\varphi_1, \varphi_2, \ldots\}$$

$$\varphi_1 = P_0 = 1 \quad \text{(degree zero)}$$

$$\varphi_2 = P_1 - \mathrm{Proj}_{\{\varphi_1\}} P_1$$

$$= P_1 - \frac{(\varphi_0, P_1)}{(\varphi_0, \varphi_0)} \varphi_0$$

$$= P_1 - \frac{\int_{-1}^{1} x \cdot dx}{\int_{-1}^{1} dx} \cdot 1$$

$$= p_1 - \underset{\text{anti-symmetry}}{\text{zero by}} = p_1 = x$$

$$\varphi_1 = x$$

$$\varphi_2 = p_2 - \frac{(\varphi_0, p_2)}{(\varphi_0, \varphi_0)}\varphi_0 - \frac{(\varphi_1, p_2)}{(\varphi_1, \varphi_1)}\varphi_1$$

$$= x^2 - \frac{\int x^2 dx}{\int dx} - \cancel{\frac{\int x^3 dx}{\int x^2 dx}} \cdot x$$

zero by antisymmetry

$$= x^2 - \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{2} \Rightarrow$$

$$\varphi_2 = x^2 - \frac{1}{3}$$

$$\varphi_n(x) = x^n - \sum_{j=0}^{n-1} \frac{\left(\int_{-1}^{1} \varphi_n(x)\varphi_j(x)\,dx\right)}{\left(\int \varphi_j^2\,dx\right)} \varphi_j(x)$$

$\uparrow$

Orthogonal polynomials on $[-1, 1]$ in the standard L_2 inner product

Legendre polynomials

(table in Wiki)

For a general $[a, b]$

$$t = \frac{x+1}{2}(b-a) + a$$

$t \in [a, b]$ when $x \in [-1, 1]$

$$0 = \int_{-1}^{1} \varphi_j(x)\varphi_k(x)\,dx = \int_a^b \varphi_j(t)\varphi_k(t)\,dt$$

$j \neq k$

$$x \longleftrightarrow t$$

Legendre polys on $[-1, 1]$ $\longleftrightarrow$ Orthogonal polys on $[a, b]$

Recall: Optimal L_2 approx:

$$f(x) \simeq p_n^*(x) = \sum_{k=0}^{n} a_k \varphi_k(x)$$

Legendre polynomials

$$a_k = \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)} = \frac{\int f(x)\varphi_k(x)\,dx}{\int \varphi_k^2(x)\,dx}$$

Homework: $f(x) = \sin(x)$ on $[0, \pi]$

E.g. $\varphi_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$

Chebyshev polynomials

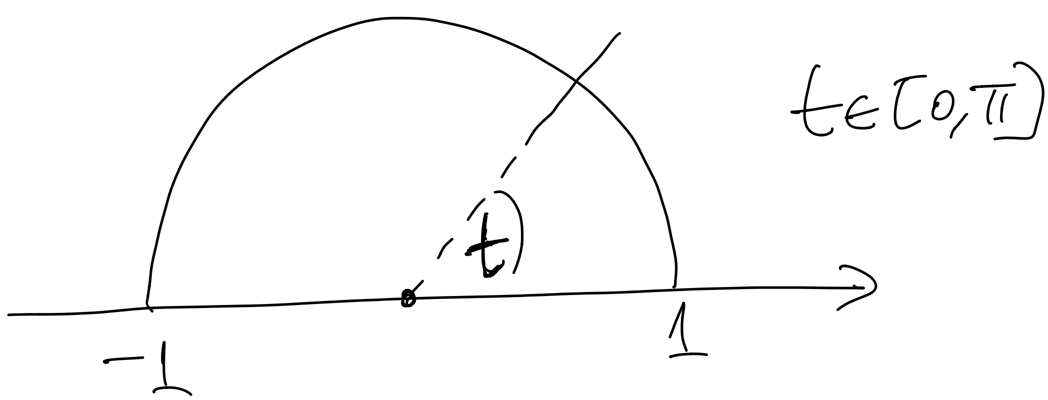
$$e^{ix} = \cos x + i \sin x$$

$$\to \int_0^{\pi} \cos(nt)\cos(mt)\,dt = 0$$

if $m \neq n$

cos's are orthogonal in L_2

on $(0, \pi)$ (trigonometric orthogonal polynomials)



$x = \cos t \in [-1, 1]$

$t = \operatorname{acos}(x)$

$$dt = \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int_{x=-1}^{1} \underbrace{\cos(m \cdot \operatorname{acos} x)}_{T_m(x)} \overbrace{\cos(n \cdot \operatorname{acos} x)}^{T_n(x)} \cdot \frac{dx}{\sqrt{1-x^2}} = 0$$

$$T_k(x) = \cos(k \cdot \operatorname{acos} x)$$

$k = 0, 1, 2, \ldots$

A polynomial

$$\begin{cases} T_0 = 1 \\ T_1 = x \\ T_2 = 2x^2 - 1 \quad \text{or} \quad x^2 - \frac{1}{2} \end{cases}$$

$T_3 = 4x^3 - 3x$ or $x^3 - \frac{3}{4}x$

...

not $x^2 - \frac{1}{3}$

Legendre

$$\int_{x=-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\left(T_m, T_n\right)_{(1-x^2)^{-1/2}} = 0$$

$$(f, g) = \int_{x=-1}^{1} \frac{f(x)\, g(x)\, dx}{\sqrt{1-x^2}}$$

Chebyshev polynomials are orthogonal w.r.t. the weighted L_2 inner product with weight function

$$w(x) = \frac{1}{\sqrt{1-x^2}} \geq 0 \quad \forall x \text{ in } [-1,1]$$

Orthogonal polys & computing

Theorem:

#1 The roots of an orthogonal polynomial $\varphi_{j\geq 1}$ are real & distinct and in (a,b)

[See thm 9.4 in theory textbook]

#2 These roots are "good" nodes for polynomial interpolation

E.g. Legendre: $x^2 - \frac{1}{3} = 0 \Rightarrow x = \pm\sqrt{\frac{1}{3}}$

Chebyshev: $x^2 - \frac{1}{2} = 0 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$

Roots of Chebyshev polys:

$$T_k(x) = \cos(k \cdot \operatorname{acos}(x)) = 0$$

$$k \cdot \operatorname{acos}(x) = (2n+1)\frac{\pi}{2}$$

$$n \in \mathbb{Z}$$

$$\operatorname{acos} x = \frac{2n+1}{k}\,\frac{\pi}{2}$$

$$\begin{cases} x_n = \cos\left(\dfrac{2n+1}{k}\,\dfrac{\pi}{2}\right) \\ \text{is a root of } T_k(x) \end{cases}$$

$$t \in (0, \pi)$$

$$x \in (-1, 1)$$

$$0 \leq \frac{2n+1}{k} \leq 2$$

$$\Rightarrow \quad 0 < n \leq \frac{(2k-1)}{2}$$

Do not include $-1, 1$

$$x_n = \cos\left(\frac{(n+1/2)}{k}\pi\right)$$

$$n = 0, \ldots, \frac{2k-1}{2}$$

or equally good

$$x_n = \cos\left(\frac{n}{k+1}\pi\right)$$