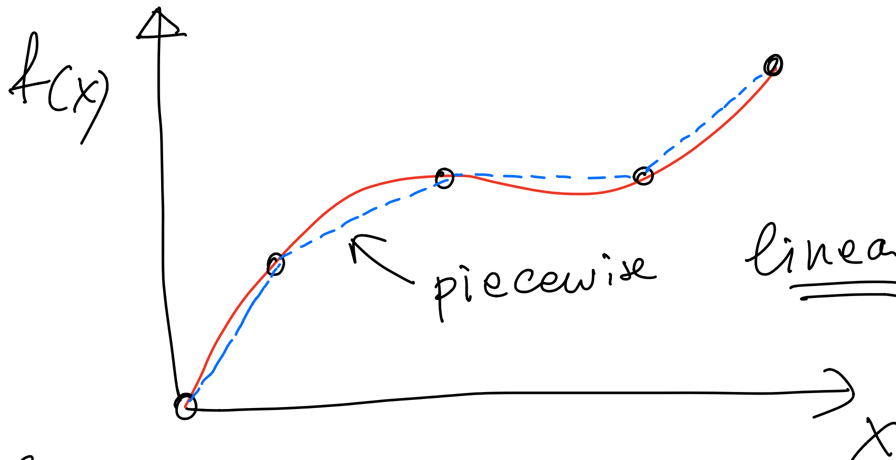
Piecewise Polynomial Interpolation

A. Donev, Spring 2021

As we saw, polynomial interpolation on n equi-spaced nodes does not converge for large n. While non-equispaced nodes can fix this, in many applications (e.g. engineering) we want to be able to use arbitrary grids.

Furthermore, we often need to work with non-smooth functions, for which large-degree polynomial approximants are inaccurate.

An alternative to global polynomial interpolation is to use piecewise (local) polynomial interpolation.



The key idea is very simple:

Break $[a,b]$ into disjoint sub-intervals. E.g. equal pieces

$$x_k = a + \frac{b-a}{n} \cdot k \ , \ k=0,\dots,n+1$$

Approximate $f(x) \simeq p_k(x)$ in each interval $[x_{k-1}, x_k]$

with a low-order polynomial $P_k(x)$. Notice this is a different local polynomial in each interval.

E.g. piecewise linear

$$P_k(x) = f(x_{k-1}) \frac{x - x_k}{x_{k-1} - x_k} + f(x_k) \frac{x - x_{k-1}}{x_k - x_{k-1}}$$

(Lagrange form of interpolant)

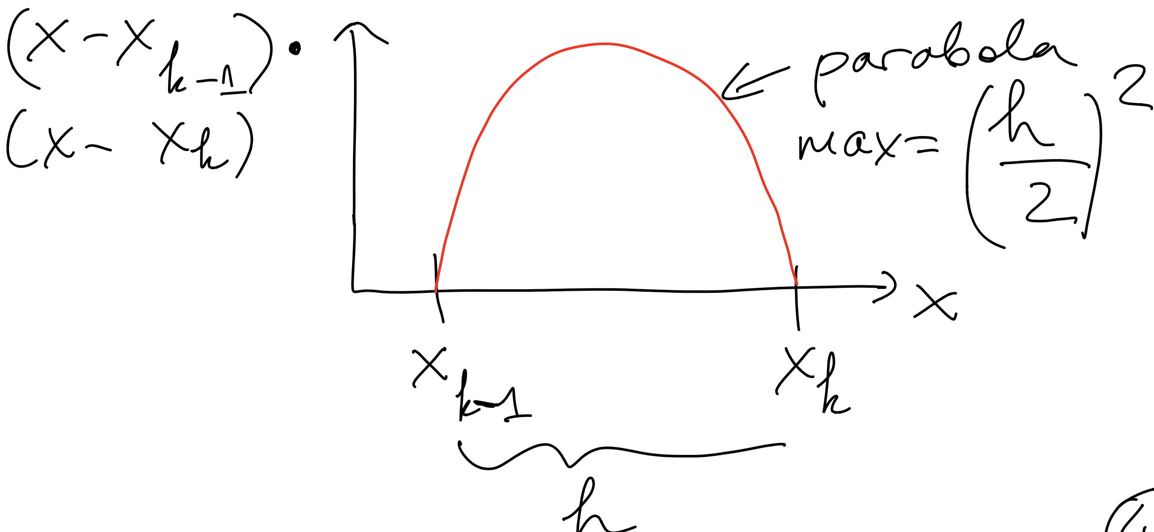
We can now piece together all of the $P_k(x)$ to form an approximation to $f(x)$ on $[a, b]$

How accurate is this piecewise linear approximation?

Since on $[x_{k-1}, x_k]$ the polynomial $p(x)$ is an interpolant, we can use the error estimate from last lecture:

$$f(x) - p_k(x) = \frac{f''(\xi(x))}{2}(x - x_{k-1})(x - x_k)$$

where $x, \xi(x) \in [x_{k-1}, x_k]$



where grid spacing

$$h = x_k - x_{k-1}$$

This gives us

$$|f(x) - P_k(x)| \leq \frac{\max_{x_{k-1} \leq x \leq x_k} |f''(x)|}{8} h^2$$

$$\text{error} = O(h^2)$$ "order of h^2"

It means that if we reduce h by a factor of 2, the error reduces by a factor of $2^2 = 4$!

So we are guaranteed second-order convergence

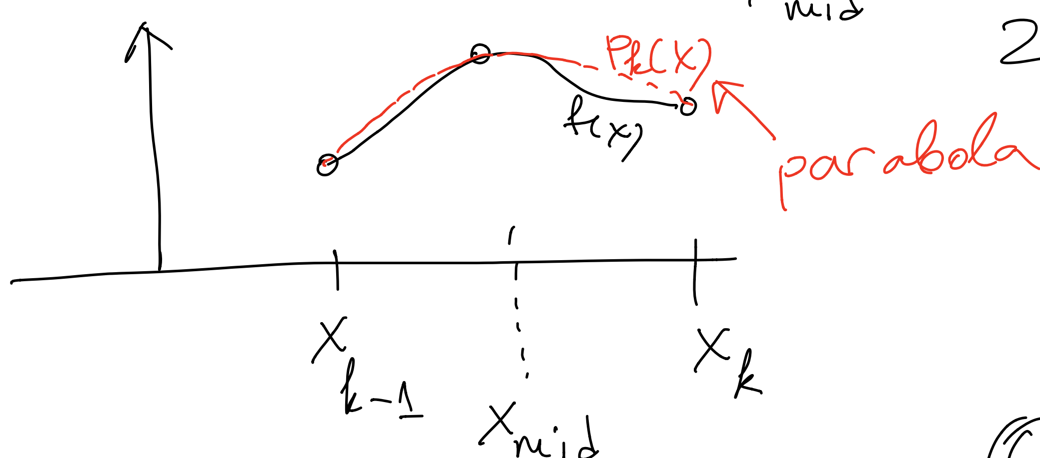
$$\left| f - \underset{\text{approx}}{\text{piecewise}} \right|_\infty = O\left(\frac{1}{n^2}\right)$$

where n is the number of intervals/points used.

No Runge phenomenon any more; we get convergence $\Rightarrow$ piecewise polynomial approximation is not very accurate but it is robust.

We can improve the order of accuracy by using a piecewise quadratic interpolant

$$x_{mid} = \frac{x_k + x_{k-1}}{2}$$



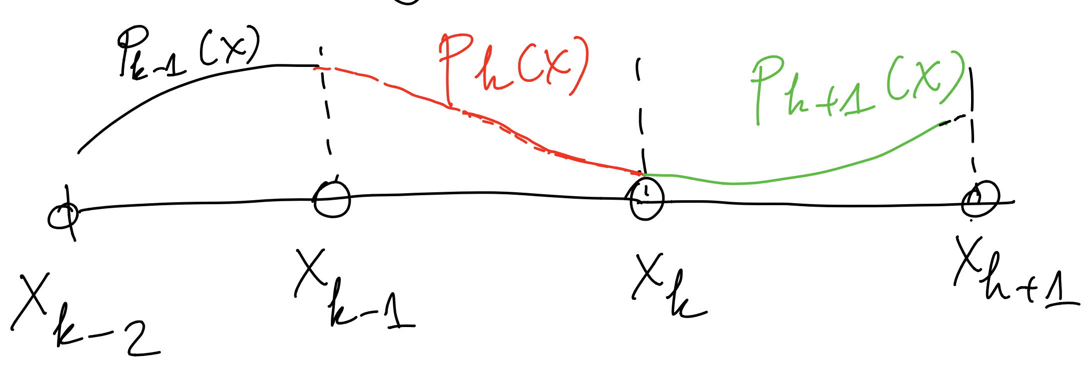
Now the error would be

$$|f(x) - p_k(x)| \sim |f'''(x)|\, h^3 = O(h^3)$$

which is <u>third-order accurate</u>

Spline interpolants

Often we want to make sure that our approximant $\tilde{f}(x) \simeq f(x)$ is smooth enough, for example, continuously differentiable.



We want to require:

$$P_k'(x_k) = P_{k+1}'(x_k) \quad \ldots (*)$$

and if we want a twice continuously-differentiable approximation then also

$$P_k''(x_k) = P_{k+1}''(x_k) \quad \ldots (**)$$

in addition to the usual interpolation condition

$$f(x_k) = P_k(x_k) = P_{k+1}(x_k) \quad \ldots (\square)$$

Let's assume that in each interval we use a quadratic polynomial $P_k(x)$.

Can we satisfy both $(\square)$ and $(*)$?

On each interval we have 3 unknown coefficients, so

$\#$ unknowns $= 3n$

On each endpoint of an interval we have 2 equations ($\square$) (one at each endpoint)

and at each interior node

($*$) adds one equation $\Rightarrow$

$\#$ equations $= 2 \cdot n + (n-1) = 3n-1$

equations.

So if we add one more equation we could solve for $3n$ unknowns

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Such quadratic spline interpolation is rarely used. Instead, more common & popular is cubic spline interpolation.

Here $P_k(x)$ is cubic so there are 4 unknown coefficients in each sub-interval, or a total of $4n$ unknowns.

If we add also condition (∗∗) in addition to ($\square$) and (∗), we get 1 more equation at each node, so

# of unknowns $= 4n$

# of equations $= 3n-1 + n-1$
$= 4n-2$

and if we add two more equations we could find a unique solution.

A common choice is to require

$$p_1''(x_0) = p_n''(x_n) = 0 \quad (\cancel{***})$$

for a <u>natural</u> cubic spline.

to get $4n$ unknowns & $4n$ equations.

<u>Summary</u>:

The natural cubic spline is the unique! C^2 (twice cont. diff.) piecewise cubic interpolant that has (approximately) minimal overall curvature.

How do we find the cubic spline interpolant?

Since $P_k(x)$ is cubic, $P_k''(x)$ is linear in x.

Denote $z_k = P_k''(x_k)$. Then

$P_k''(x)$ must be equal to z_k at nodes, and be linear in each sub-interval $\Rightarrow$

$P_k''(x)$ is the piecewise linear interpolant.

$$P_k''(x) = z_{i-1} \frac{x - x_k}{x_{k-1} - x_k} + z_i \frac{x - x_{k-1}}{x_k - x_{k-1}}$$

in $[x_{k-1}, x_k]$

Simplify algebra (not required) by assuming equi-spaced nodes, and integrate twice to get

$$P_k(x) = \frac{1}{h} z_{k-1} \frac{(x_k - x)^3}{6} +$$

$$\frac{1}{h} z_k \frac{(x - x_{k-1})^3}{6}$$

$$+ C_k (x - x_{k-1}) + D_k$$

unknown coefficients/
integration constants

But we have the conditions:

(I) at x_{k-1}:

$$\frac{1}{h} \frac{h^3}{6} z_{k-1} + D_k = f_{k-1} = f(x_{k-1})$$

$$\Rightarrow D_k = f_{k-1} - \frac{h^2}{6} z_{k-1}$$

(D) at x_k:

substitute

$$\frac{h^2}{6} z_k + C_k h + D_k = f_k$$

$$\Rightarrow C_k = \frac{1}{h}\left[(f_k - f_{k-1}) + \frac{h^2}{6}(z_{k-1} - z_k) \right]$$

Now we have a formula for P_k in terms of the unknown z_k's. To determine z_k's, we use (*) and (**) at each interior node, and (***) at the two end points ($z_0 = z_n = 0$)

to obtain a tridiagonal linear system for the unknown's

$$\begin{bmatrix} 2/3 & 1/6 & & \\ 1/6 & \ddots & \ddots & \\ & \ddots & \ddots & 1/6 \\ & & 1/6 & 2/3 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-2} \end{bmatrix}$$

symmetric

$$b_k = \frac{1}{h^2}\left(f_{k+1} - 2f_k + f_{k-1}\right)$$

In a worksheet / homework you will learn that

$$b_k \simeq f''(x_k)$$

which makes sense because $z_k \simeq f''(x_k)$ is a second derivative

Also in homework you obtained an $O(n)$ simple algorithm to do LU factorization of tridiagonal matrices. So we can compute the cubic spline fast even for millions of points (e.g. computer graphics) without a problem

In Matlab, use spline to compute spline approximant, and ppval to evaluate it

Cubic splines approximate $f(x)$ to third order, $f'(x)$ to second order, and $f''(x)$ to first order.