Singular values of matrix

"tells you everything about A"

$$A = U \Sigma V^*$$

"ultimate" decomposition

diagonal

unitary

works for any matrix A!

$$[m \times n] = [m \times m][m \times n][n \times n]$$

Singular value decomposition

Columns of U are left singular vectors

Columns of V are right sing. vectors

Diag. elements of Σ are called singular values

$$\Sigma = m \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \ddots & & \vdots & 0 \\ 0 & & \sigma_m & \vdots & \end{bmatrix} \quad \text{If } m < n$$

$$\qquad\quad m \qquad\qquad n$$

$$\Sigma = \begin{array}{c} n \\ \begin{bmatrix} \sigma_1 & \cdots & \sigma_n \\ \hline & 0 & \end{bmatrix} \end{array} n \quad \text{If } m > n$$

$$\sigma_1, \ldots\ldots, \sigma_p \geq 0$$

$$\sigma_{p+1}, \ldots\ldots = 0$$

$$p = \min(m, n)$$

<u>Observe</u>

If A is Hermitian

$$\begin{cases} A = U \Lambda U^* \\ \quad\;\; \| \;\; \equiv \;\; \| \\ A = U \Sigma V^* \end{cases} \qquad \begin{array}{l} U = V \\ \Sigma = \Lambda \end{array}$$

For symmetric matrices
no difference between eigenvalue
real
& singular value decomposition.

SVD works for all
matrices & can be
computed numerically in
a stable way

symmetric

$$AA^* \quad \text{or} \quad A^*A$$

conj. transpose.

$$(AA^*)^* = (A^*)^* A^* = AA^*$$

$$(A^*A)^* = A^*A$$

Idea: Compute eigenvalue decomposition of either A^*A or AA^* (unitarily diagonalizable)

$$A = \underbrace{U \Sigma V^*} \quad \text{(SVD)}$$

$$A^*A = (U\Sigma V^*)^*(U\Sigma V^*)$$

$$= V\Sigma \underbrace{(U^* U)}_{\text{Identity matrix}} \Sigma V^*$$

$$= V\Sigma^2 V^*$$

$$\underbrace{AA^*} = \underbrace{U \Sigma^2 U^*}_{\text{eigenvalue decomp. of } AA^*}$$

Columns of U are eigenvectors of AA^*, & columns of V are eigenvectors of A^*A

$$\Sigma^2 = \Lambda$$

$$\sigma_i^2 = \lambda_i \geq 0$$

eigenvalues of AA^* or A^*A

[sidenote] AA^* & A^*A are symmetric positive semidefinite matrices

Define: $\sigma_i = \sqrt{\lambda_i}$

Importance of SVD

Remember:

$$\|A\|_{1,2,\infty} = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$= \sup_{\|x\|=1} \|Ax\|_{1,2,\infty}$$

$$\underbrace{\|A\|_2}_{\text{spectral norm}} = \underbrace{\sup_{\|x\|_2=1} \|Ax\|_2}_{?}$$

aside

Frobenius norm

$$\|A\|_F = \sqrt{\sum |a_{i,j}|^2}$$

$$\neq \|A\|_2$$

(Not reduced because
$\|I\|_F = \sqrt{n} \neq 1$)

$$\sup_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{(Ax)^T (Ax)}{x^T x}$$

$$\boxed{\|x\|_2^2 = x \cdot x = \sum x_i^2 = x^T x}$$

$$\|A\|_2^2 = \sup_{x \neq 0} \frac{x^T (A^T A) x}{x^T x}$$

Rayleigh quotient $\left\{ \sup_{x \neq 0} \frac{x^T M x}{x^T x} \right.$, $M = A^T A$

$= \lambda_{max}(A^T A)$

$= \sigma_1^2$ (largest σ)

$$\boxed{\|A\|_2 = \sigma_1}$$

$$K_2(A) = \|A\|_2 \|A^{-1}\|_2$$

Eigen/singular values of A^{-1} are the inverses of the eigen/singular values of A

$$\sigma_{max}(A^{-1}) = \sigma_{min}(A)$$

$$\Rightarrow \quad K_2(A) = \frac{\sigma_{max}}{\sigma_{min}} \geq 1$$

SVD in MATLAB

$$A = U \Sigma V^*$$

$$[m \times n] = [m \times m][m \times n][n \times n]$$

If $m >> n$ or
$n >> m$

$$\left.\begin{aligned} m \times m &= O(m^2) \\ m \times n &= O(n^2) \end{aligned}\right\} \text{memory}$$

Reduced (economy) SVD

$$A = U \Sigma V^*$$

$$[m \times n] = [m \times n][n \times n][n \times n]$$

$m > n$

Or if $m < n$

$$[m \times n] = [m \times m][m \times m][m \times n]$$

$$\sigma_1, \sigma_2, \ldots, \sigma_p \geq 0$$

singular values

$$\sigma_{p+1}, \sigma_{p+2}, \ldots = 0$$

$$p = \min(m, n)$$

$$p = \mathrm{rank}(A)$$

number of nonzero singular values

SVD is a rank-revealing factorization

MATLAB:

```
[U, Σ, V] = svd(A, 'econ')
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Uses / properties of SVD

$$A = U \Sigma V^*$$

$$Ax = 0 \Rightarrow x \in \text{null}(A)$$

$$U^* \,|\, U \Sigma V^* x = 0$$

U, V are square & invertible

$$\begin{cases} V^{-1} = V^* \\ U^{-1} = U \end{cases}$$

$$\Sigma \underbrace{V^* x} = 0$$

$$\Sigma y = 0$$

$$\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_p \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = 0 = \begin{bmatrix} \sigma_1 y_1 \\ \vdots \\ \sigma_p y_p \end{bmatrix}$$

$$\sigma_i y_i = 0$$

Either

$\sigma_i = 0$

or $y_i = 0$

or both

$\Rightarrow$ Only values of y corresponding to <u>zero</u> singular values can be nonzero

$$y = V^* x$$

$$x = (V^*)^{-1} y = (V^*)^* y$$

$$x = V y$$

x is a linear combination of columns of V corresponding to <u>zero</u> singular values

$\{v_{r+1}, \ldots, v_n\}$ form a orthonormal basis for null (A)

$$\{Ax\} = \text{Image}(A)$$
$$\text{range}(A)$$
$$\text{col}(A)$$

$$U\Sigma(V^* x) = \underbrace{U\Sigma y} =$$

$$= U \begin{bmatrix} \sigma_1 y_1 \\ \vdots \\ \sigma_r y_r \\ 0 \end{bmatrix}$$

$\{u_1, \ldots, u_r\}$ are an orthonormal basis for range(A)

Pseudo-inverse of matrix

If A is invertible

$$A = U \underset{\substack{\vee \\ 0}}{\Sigma} V^*$$

$$A^{-1} = (V^*)^{-1} \Sigma^{-1} U^{-1}$$

$$A^{-1} = V \Sigma^{-1} U^*$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ & & & \sigma_n^{-1} \end{bmatrix}$$

$$x = A^{-1} b = V\left(\Sigma^{-1}\left(U^* b\right)\right)$$

solve a linear system

Define

$$\Sigma^{\dagger} = \begin{bmatrix} \sigma_1^{-1} & & & \\ & \ddots & & \\ & & \sigma_r^{-1} & \\ & & & \phi \\ & & & & \ddots \end{bmatrix}$$

$$A^{\dagger} = U \Sigma^{\dagger} V^{*}$$

matrix pseudo-inverse (also for non-square matrices)

Moore-Penrose inverse

$$Ax = b$$

[A can be non-square]

$$x = A^{\dagger} b$$

← pinv in MATLAB

Numerically, if

$$\sigma_i < [10^{-16} - 10^{-14}] \cdot \sigma_1$$

Then we can set $\sigma_i = 0$

If $\sigma_i < \varepsilon \cdot \sigma_1$ set $\sigma_i = 0$ } pinv

tolerance $10^{-12} - 10^{-6}$