Worksheet 1 (Feb 18th, 2021)

Steffensen's Method

The goal of this task is to analyze Steffensen's Method (see also problem 1.4 in theory textbook, and problem 16 in chapter 4 in Practice textbook) for finding the root of a function $f : \mathbb{R} \to \mathbb{R}$, defined by the iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{\lambda_k}, \qquad k = 1, 2, \ldots. \tag{1}$$

where

$$\lambda_k = \frac{f\left(x_k + f(x_k)\right) - f(x_k)}{f(x_k)}.$$

1. Explain how this iteration relates to Newton's method and the Secant method. Is it closer to one or the other as $x_k \to x$ where $f(x) = 0$? Based on this, how fast would you expect it to converge, linearly or quadratically? [*Hint: No right or wrong answer here, this is just to get you thinking.*]

2. Implement this method in Matlab (do it together with one person sharing Matlab screen, but everyone should comment and try to improve the code). Feel free to use one of the codes from class as a place to start and do your best to write the code in a way that works for any $f(x)$. Stop iterating when $|f(x_k)| < \epsilon$ (this is called the "termination criterion"), where $\epsilon > 0$ is an input tolerance.

3. Use the method as a way to compute $\sqrt{c}$ by solving $x^2 - c = 0$ for $c = 13$. Try the method for several initial guesses in the interval $[0, c]$ and report whether it converges.

4. What does λ_k converge to as $x_k \to x$ converges to the root for a general function $f(x)$? Can you confirm this numerically for the square root problem? How many digits of accuracy did you get for $\lim_{k\to\infty} \lambda_k$ depending on the tolerance you use? Are you loosing digits and why? [*Hint: You may run into roundoff error.*]

5. [**Finish on your own** at home if no time in class] Does Steffensen's method converge slower or faster than the Babylonian method? Try to make a plot comparing the two on the same graph, and make sure the figure is well-labeled and readable. Remember that quadratic convergence means that
$$\frac{|e_{k+1}|}{e_k^2} \to C = \text{const}.$$
Does Steffensen's method converge quadratically? Can you determine the constant C for the computation of $\sqrt{13}$? [*Hint: You may run into roundoff error.*]

6. [**Optional** work on your own or feel free to stay longer after class, but try on your own at home first!] There are a number of different ways to analyze the convergence Steffenson's method, notably, one can follow what we did for Newton's method. Another way is to use the fixed-point iteration approach, $x_{k+1} = g(x_k)$. What is $g(x)$ for this method? Can you compute its derivative

$g'(x)$ [*Hint: Master the chain rule.*] We want to evaluate this at the root itself, but this requires carefully taking the limit $x_k \to x$. To do this, introduce some shorthand notation:

$$f(x) = h, \quad \Delta f = f(x+h) - f(x).$$

We want to take the limit $h \to 0$, using L'Hopital's rule, or, equivalently, using Taylor series of things like Δf in terms of powers of h. Try to do as many steps of this calculation as you can. The final answer is $g'(x) = 0$; what does this imply for the convergence of this method?
Note: If you cannot do this for a general $f(x)$, at least do it for the specific $f(x) = x^2 - c$ where the calculation is just simple algebra, and confirm that $g'(x) = 0$ at the root.