Worksheet 2 (Feb 24th, 2021)

Newton's method in higher dimensions

Let $h : \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by $\boldsymbol{h}(x,y) = (f(x,y), g(x,y))^T$, where

$$f(x,y) = y - \frac{\sqrt{3}}{2}x^2, \quad g(x,y) = 2x^2 + 8y^2 - 8.$$

We want to find the roots of $\boldsymbol{h}$, i.e., all pairs $(x,y) \in \mathbb{R}^2$ such that $\boldsymbol{h}(x,y) = (0,0)^T$.

1. Sketch or plot the curves/sets $\mathcal{F} = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$ and $\mathcal{G} = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$, i.e., the set of all zeros of f and g. What geometrical shapes do these sets have?
2. Calculate analytically the roots of $\boldsymbol{h}$, i.e., the intersection of the sets $\mathcal{F}$ and $\mathcal{G}$.
3. Write down a first-order Taylor series expansion for $\boldsymbol{h}$ (i.e., for both f and g) around a given point (x_0, y_0). Then rewrite it in the matrix-vector form

$$\boldsymbol{h}(x,y) = \boldsymbol{h}(x_0,y_0) + \boldsymbol{J}(x,y) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

   and report the *Jacobian matrix* $\boldsymbol{J}(x,y)$.
4. Recall that Newton's method computes a new guess by using the first-order Taylor series to approximate the function and setting that to zero to find the next solution. Use this to write down a linear system of equations to get the next guess $(x_{k+1}, y_{k+1})^T$ from the current guess (x_k, y_k).
5. Implement Newton's method for $\boldsymbol{h}(x,y) = (0,0)^T$ and use it to calculate the first 5 iterates for the starting values $(x_0, y_0) = (2,3)$ and $(x_0, y_0) = (-1.5, 2)$, and compare to the solution from part 2. Plot these iterates in the xy-plane together with the curves $\mathcal{F}$ and $\mathcal{G}$, and explain how you solved the linear system of equations from part 4.

As time permits or work on your own to really prepare for future graduate work in mathematics/computing:

- Use the previous question to write a general formula for the Jacobian matrix of a vector-valued function of many variables, $\boldsymbol{f}(\boldsymbol{x} \in \mathbb{R}^n) \in \mathbb{R}^m$. When $m = 1$ the Jacobian is also called the *gradient* of $\boldsymbol{f}$.
- Write down the linear system from part 4 for a general square nonlinear system of equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$.
- [This part requires the pre-recorded lecture from this week, so come back to it later:] Write the code for part 5 above so that it works in any dimension, that is, write a function that takes in as input a function $\boldsymbol{f}(\boldsymbol{x})$ and a function to compute its Jacobian matrix $\boldsymbol{J}(\boldsymbol{x})$ and an initial guess, and then does Newton's method from there. Try it for the specific two-dimensional problem.