Worksheet 3 (March 3rd, 2021)

Solving $Ax = b$ and LU factorization

1. Let's compute the LU-factorization of $A := \begin{bmatrix} 3 & 3 & 0 \\ 6 & 4 & 7 \\ -6 & -8 & 9 \end{bmatrix}$ using the following direct approach to find L and U:
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = A.$$
By multiplying appropriate rows and columns, find the entries of L and U in the following order: $u_{11}, u_{12}, u_{13}, l_{21}, l_{31}, u_{22}, u_{23}, l_{32}, u_{33}$.
Check your result somehow (this step is crucial in Numerical Analysis and will be asked on exams and homeworks frequently).

2. Use the LU factorization to solve the linear system $Ax = b$ with $b = [1, 0, 0]^\top$ using one forward and one backward substitution.
Note: This is one of the steps you will need in HW2.P2.a.

3. In the matrix A defined above, replace the $(2, 2)$-entry by 6 and again compute the LU-factorization. What do you observe? What is the rank of the new matrix?

4. Use the LU factorization to compute the determinant of A. Recall that for two matrices of appropriate sizes, $\det(AB) = \det(A)\det(B)$.
Check your result somehow.

5. Go back to the code *MyLU.m* on the course homepage and compute for each line in the code how many floating-point operations it performs. Write it down as a comment in the Matlab code.

6. Use the results from part 4 to write down a mathematical expression (sum) giving the total number of FLOPS in the LU factorization of an arbitrary $n \times n$ matrix *without* pivoting.
On your own: How much cost does (row or column) pivoting add, roughly speaking? Complete (both row *and* column) pivoting is different – can you understand why?

7. We are usually only interested in the "leading" term, i.e., the highest power of n as that term dominates the number of flops for large n. Compute that term?
Hint: You can replace the sum by an integral, because
$$\int_0^n x^p dx \leq \sum_{k=1}^{n} k^p \leq \int_1^{n+1} x^p dx,$$
see Fig. 7.1 in practice textbook.