Worksheet 7 (April 14th, 2021)

1 More on Polynomial Interpolation

In this worksheet you will explore some topics in polynomial interpolation. Each student should work on their own for a bit and then you can discuss and compare results, and come up with testing/validation strategies to see whose answer is correct as a group.

1.1 Approximating sine

Consider approximating $f(x) = \sin(\pi x)$ on $x \in [0, 1]$.

1.1.1 Lagrange Interpolation

Write down the interpolating polynomial of degree 2 (parabola) with equi-spaced nodes.

1.1.2 Hermite Interpolation

We want to find a polynomial $p(x)$ of as low a degree as possible such that the polynomial matches the value of the function *and* of the derivative at the end-points, $p(0) = f(0)$, $p(1) = f(1)$, $p'(0) = f'(0)$, $p'(1) = f'(1)$. Find such a Hermite polynomial approximant and then plot it on the same graph with the approximant from part 1.1, along with the true function. As a group, decide whether the plot makes sense and you are confident in your answers based on the plot.
Important: Make sure to understand whether what you did would work for any function and for any interval $[a, b]$ and not just for the specific example of sine on $[0, \pi]$, and if not, find a procedure that would work in general (as this more general case may appear on a homework/exam).

1.2 Approximating derivatives and integrals

Consider a function $f(x)$ such that you know it at three equi-spaced nodes, for example, $f(-h) = f^-$, $f(h) = f^+$, and $f(0) = f^0$. If we approximate the function with an interpolating polynomial, then we can differentiate that polynomial to approximate the derivative of the function. Similarly, to integrate the function we can integrate the polynomial instead. Explain (as a group) why there is no difference if I change the x position of the middle point, i.e., if I specify $f(x - h) = f^-$, $f(x + h) = f^+$, and $f(x) = f^0$; I only set $x = 0$ to simplify the algebra.

1. Use this approach to approximate $f'(0)$ and $f''(0)$ from the three values f^-, f^0 and f^+. Find some way to validate your answer (as a group).

2. Use this approach to approximate $I = \int_{-h}^{h} f(x)dx$ from the three values f^-, f^0 and f^+. Find some way to validate your answer (as a group).

3. Use the formula from part 1.2.1 to approximate $\sin''(\pi/4)$ for $h = 2^{-p}$ with $p = 3, 4, 5$,6 and report the error you get compared to the true derivative. Do you see any patterns?

4. (Optional) Another approach to approximating derivatives is to first approximate $f(x)$ with its Taylor series,
$$f(x) \approx \tilde{f}(x) = f^0 + f'(0)x + \frac{1}{2}f''(0)x^2,$$
and then obtain $f'(0)$ and $f''(0)$ from the equations $\tilde{f}(-h) = f^-$, $\tilde{f}(h) = f^+$. What does this approach give?