Finite Element Methods

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These lecture notes are based on lecture notes by Georg Stadler. I consider the background on piecewise smooth local basis functions to be part of interpolation in 2D/3D so see those lecture notes first.

Like pseudospectral methods, FEM is a series method, meaning that the discrete solution is a function that is a sum of basis functions and the discrete unknowns are the series coefficients:

$$u_h(x) = \sum_{i=1}^{N} U_i \varphi_i(x) \simeq u(x)$$

"grid size" (discrete) — unknown coefficients — basis functions

A key difference is that now the basis functions $\varphi_i(x)$ are piecewise polynomials with localized support — this will be key for efficiency as it will lead to sparse matrices not dense like for orthogonal polynomials.

But the heart of FEM methods is their relation to weak & variational formulation of elliptic (parabolic) PDEs

Consider PDE on bounded domain $\Omega \in \mathbb{R}^n$ with Lipshitz boundary $\partial\Omega$:

$\partial_j \equiv \partial / \partial x_j$

$$-\sum_{i,j=1}^{n} \partial_j \left(a_{ij}(x)\, \partial_i u\right) \equiv -\nabla\cdot\left(A(x)\nabla u\right)$$

$$+\sum_{i=1}^{n} b_i(x)\, \partial_i u \equiv b(x)\cdot\nabla u$$

$$+\ c(x)\, u = f(x)$$

Importantly, $A(x)$ is uniformly (symmetric) positive definite, i.e., $\nabla\cdot(A\nabla u)$ is an elliptic operator.

BCs can be Dirichlet (u), Neumann ($\partial u / \partial n$), or Robin/mixed.

If $b = 0$ (no "advection"), we have a variational formulation of PDE. Take for simplicity

$$\begin{cases} -\nabla\cdot(A\nabla u) + u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega_1 \quad \text{(essential BC)} \\ a\dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega_2 \quad \text{(natural BC)} \end{cases}$$

Take a test function $v \in C^1(\Omega)$ with $v|_{\partial\Omega_1} = 0$ (essential BCs must be incorporated into FEM spaces / enforced explicitly in the strong sense), multiply the PDE and integrate by parts to lower smoothness requirements

$$-\int_\Omega \nabla\cdot(a\nabla u)\, v\, dx + \int_\Omega u v\, dx =$$

$$= \int_\Omega a \nabla u\cdot\nabla v\, dx - \int_{\partial\Omega} a \frac{\partial u}{\partial n} v\, ds$$

$$+ \int_\Omega u v\, dx = \int_\Omega f v\, dx$$

Using BCs we get

$$\int_\Omega a \nabla u\cdot\nabla v\, dx + \int_\Omega u v\, dx =$$

$$\int_\Omega f v\, dx + \int_{\partial\Omega_2} g v\, dx \quad \ldots (*)$$

Weak formulation: $(*)$ is true for all suitable $v(x)$

The right function space is the same for u and v (for self-adjoint problems, but in Petrov-Galerkin methods u and v belong to <u>different</u> spaces) is

$$H^1_{0,\partial\Omega_1}(\Omega) = \Big\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \ \forall i = 1, \dots, n, \quad u = 0 \text{ on } \partial\Omega_1 \Big\}$$

Sobolev space $H^1 \equiv W^{1,2}$

Denote bilinear form

$$a(u,v) = \int_\Omega a(x)\, \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx$$

and linear form

$$\ell(v) = \int_{\Omega} f v \, dx + \int_{\partial\Omega_2} g v \, ds$$

Variational / weak form of PDE:

$$\begin{cases} a(u,v) = \ell(v) \quad , u \in H^1_{0,\partial\Omega_1} \\ \forall v \in H^1_{0,\partial\Omega_1} \end{cases}$$

If $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega_2)$

then Lax-Milgram lemma says u is a unique solution.

Key condition is coercivity/ellipticity:

$$a(v,v) \geq c_0 \, \|v\|_{H^1}$$

$$(v,w)_{H^1} = \int_{\Omega} \left(\nabla v \cdot \nabla w + v \cdot w \right) dx$$

If $A(x)$ is SPD for all x,

$$a(u,v) = a(v,u)$$

we have also equivalent energy/variational formulation

$$u = \arg \min_{v \in H^1_{0,\partial\Omega_1}} J(v)$$

$$J(v) = \frac{1}{2} a(v,v) - \ell(v)$$

Steps in FEM

1) Write weak form of PDE
2) Choose finite dimensional spaces for all function spaces
3) Solve resulting system of equations

So instead of

Find $u \in V$ s.t. $a(u, v) = \ell(v)\ \forall v$

choose finite-dimensional $V_h \subset V$ made of piecewise polynomial functions and solve

$$\boxed{\begin{array}{l}\text{Find } u_h \in V_h \text{ s.t.} \\ a(u_h, v_h) = \ell(v_h) \quad \forall\, v_h \in V_h\end{array}}$$

by solving a system of equations.

$$V_h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_N\}$$

(linearly independent but not orthogonal)

$$u_h = \sum_{i=1}^{N} U_i\, \varphi_i(x)$$

Plug into weak form to get

$$\sum_{i=1}^{N} a(\varphi_i, \varphi_j)\, U_i = \ell(\varphi_j) \quad \forall j = 1, \dots, N$$

$$\begin{cases} AU = L & \text{– system of } N \text{ equations} \\ A_{ij} = a(\varphi_i, \varphi_j) & \text{stiffness matrix} \\ L_i = \ell(\varphi_i) \end{cases}$$

Notes :

1. Since computing A_{ij} requires integration, it may have to itself be approximated by spectral quadrature (e.g. Gauss quad).
   Always true for r.h.s. L

2. By choosing piecewise basis wisely we can make A be sparse & SPD and thus solve system more efficiently

Parabolic problems (aside)

$$\begin{cases} \frac{\partial u}{\partial t} + a u_x = d u_{xx} + b u + f(x,t) \\ u(\partial\Omega) = 0 \end{cases}$$

constant coeff.

Method of lines:

$$u(x,t) = (u(t))(x)$$

$$u : (0,T) \to H^1_0(\Omega)$$

Weak form:

$$\int_\Omega v u_t \, dx = \int_\Omega v(-a u_x + d u_{xx} + b u + f) \quad \forall v \in H^1_0(\Omega)$$

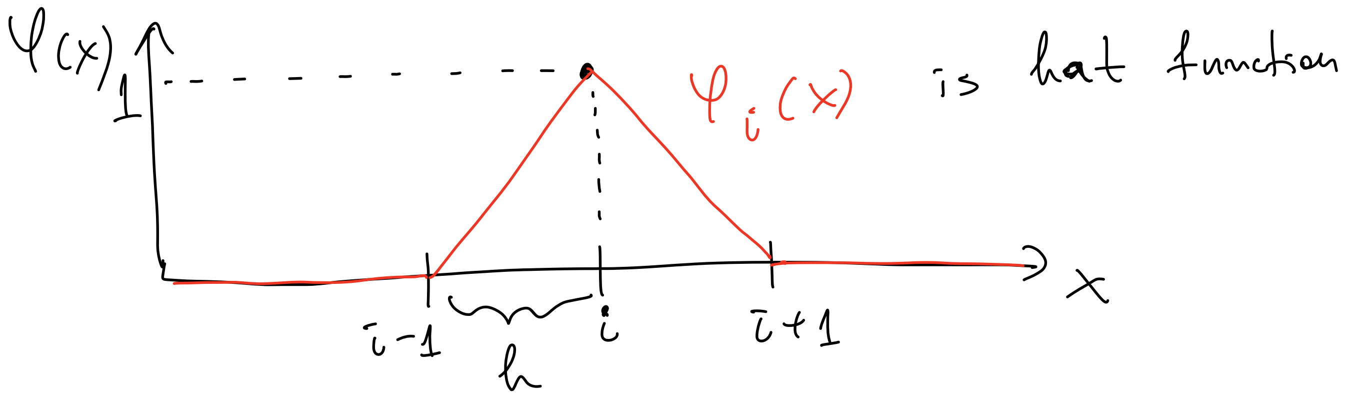
integrate by parts

$$u(x,t) = \sum_{i=1}^{N} U_i(t) \varphi_i(x)$$

gives $M \frac{dU}{dt} + AU = g$ (ODEs)

mass matrix $\quad M_{ij} = (\varphi_j, \varphi_i)_{L^2(\Omega)}$

Take uniform grid in 1D



$$\Rightarrow \int \varphi_j \varphi_{j+1} \, dx = \frac{h}{6}$$

$$\int \varphi_j^2 \, dx = \frac{2}{3} h$$

$$-\int \frac{d\varphi_{j-1}}{dx} \varphi_j \, dx = 1/2$$

$$\int \left(\frac{d\varphi_j}{dx}\right)^2 dx = \frac{2}{h^2}$$

$$\int \left(\frac{d\varphi_{j-1}}{dx}\right)\left(\frac{d\varphi_j}{dx}\right) dx = -\frac{1}{h^2}$$

Gives discretization

$$M \frac{dU}{dt} + a\tilde{D}U = d D_2 U + b M U + \mathbf{F}$$

where $M = \frac{1}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & 1 \\ & \ddots & 1 & 4 \end{bmatrix}$ mass matrix

$$\tilde{D} = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix} = \text{centered difference}$$

$$D_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & 1 \\ & \ddots & 1 & -2 \end{bmatrix} = \text{standard Laplacian}$$

$$F_i = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} f(x)\,\varphi_i(x)\,dx$$

Except for mass matrix, this is the same as the FD second order!

We know that centered difference is not good for advection (will require RK3+ to integrate).

But FEM can be higher order (with some conditioning issues) & use unstructured grids.

Note that

$$M^{-1}\tilde{D}\,U \simeq \frac{\partial u}{\partial x} + O(h^4)$$

in the finite difference sense (called "compact finite difference")

So in practice the method will be better than 2^{nd} order FD for advection. But each timestep requires solving $Mx=b$!

Lumped mass approximation: Approx. M by a diagonal matrix

Back to time-independent problems

We will not go into the extensive & well-developed theory of FEM methods, which relies heavily on Sobolev function spaces. Some notes:

① Cea Lemma:

The FEM solution is nearly optimal in the approximation space:

$$\|u - u_h\|_{H^1} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_{H^1}$$

As long as the constants c_1 and c_0 are well-behaved, and the approximation is suited to the PDE, we don't have to worry & have strong theoretical guarantees.

(2) The target in the FEM world is to prove a priori error bound

$$\|u - u_h\|_{H^1} \leq C h^p$$

where p is the degree of the polynomial basis functions (so linear gives first order convergence in H^1 in general)

(3) For purely elliptic PDEs, define inner product

$$(v, w)_a = a(v, w)$$

From

$$\begin{aligned} &\text{PDE} \quad a(u, v_h) = \ell(v_h) \\ &\text{FEM} \quad a(u_h, v_h) = \ell(v_h) \end{aligned} \quad \forall v_h \in V_h$$

$$\Rightarrow \quad a(u - u_h, v_h) = 0$$

$\Rightarrow$ Error is orthogonal to V_h in the new inner product, i.e.

$$\|u-u_h\|_a = \min_{v_h \in V_h} \|u-v_h\|_a$$

FEM approximation is optimal in the a-norm (improved Cea lemma)

For example, for

$$\begin{cases} -u'' + u = f & \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

and a regular 1D grid using:

Cea's lemma + interpolation error bound + elliptic regularity ($\|u\|_{H_2} \leq C\|f\|_{L_2}$) one gets:

$$\|u-u_h\|_{H^1} \leq \frac{2h}{\pi}\left(1+\frac{h^2}{\pi^2}\right)^{1/2}\|f\|_{L_2}$$

However, since we know that for regular grids + linear basis FEM is the same as FD 2^{nd} order, we expect that the solution is more accurate than just 1^{st} order. For this one needs to switch to a different norm that does <u>not</u> test derivatives since those <u>are</u> indeed only first-order accurate. Specifically, one can show

$$\|u - u_h\|_{L_2} \leq 4 h^2 |u|_{H^2}$$

i.e. solution is second-order accurate in L_2 norm.

However, FEM error bounds can become useless if the approximation space is <u>not</u> suited to the PDE.

Notably, for advection-diffusion:

$$\begin{cases} -d\,\nabla^2 u + \vec{a}\cdot\nabla u = f \\ \nabla\cdot\vec{a} = 0 \end{cases}$$

the standard FEM discretization gives an error constant

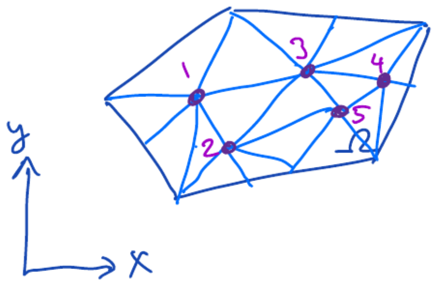
$$C \sim \sqrt{1 + P_e^2}$$

where P_e is the Pechlet number. So for <u>advection-dominated</u> problems $C \gg 1$ and FEM does not work well without some "stabilization"

Some practicalities:

FEM Grids & Matrices

$\Omega \subset \mathbb{R}^2$ polygonial boundary, Cover Ω with triangles



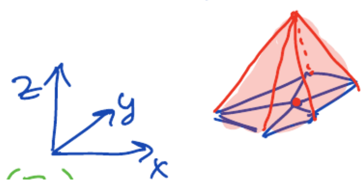
• interior points

V_h ... space of continuous functions that are linear on each triangle

$V_h \subset V$

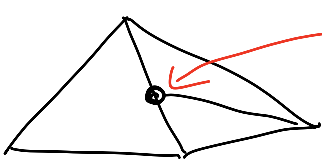
ϕ_i basis for each interior node, $i = 1, \dots, 5$

2-dim. hat functions:



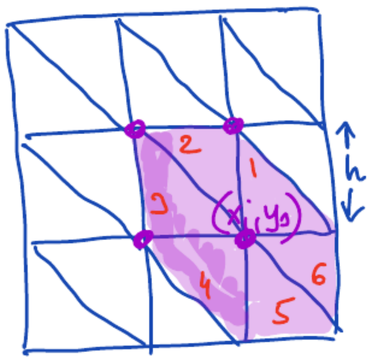
$\phi_i(x,y)$

In 2D, almost any domain of interest can be triangulated, so take FEM cells to be triangles, FEM nodes to be the vertices, no hanging nodes



not OK in standard FEM

If $\Omega = [0,1]^2$ unit square with uniform triangulation



$\phi_{ij}(x,y)$

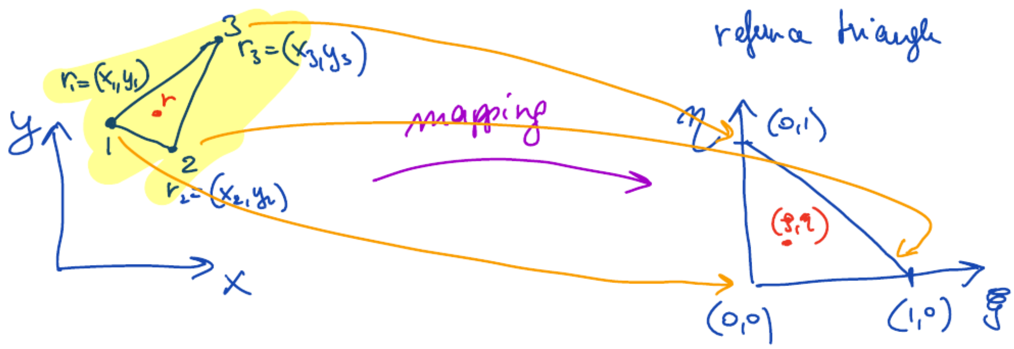
Piecewise linear tent functions, give $U_i = u_h(x_i)$ as in FD

For the Laplacian,

$$A_{ij} = \int \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy$$

is the standard FD 5^{pt} Laplacian, and so just as ill-conditioned as for FD methods: Efficient linear solvers are iterative & based on geometric or algebraic multigrid method (AMG)

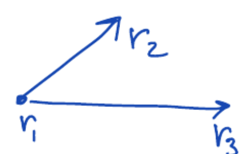
In FEM, typically things are precomputed for a reference triangle, and results are mapped to each triangle of the grid using suitable Jacobians.



$$r=(x,y) = \underbrace{(1-\xi-\eta)}_{\varphi_1(\xi,\eta)} r_1 + \underbrace{\xi}_{\varphi_2(\xi,\eta)} r_2 + \underbrace{\eta}_{\varphi_3(\xi,\eta)} r_3$$

Consider trafo: $(\xi,\eta) \longmapsto r=(x,y)$

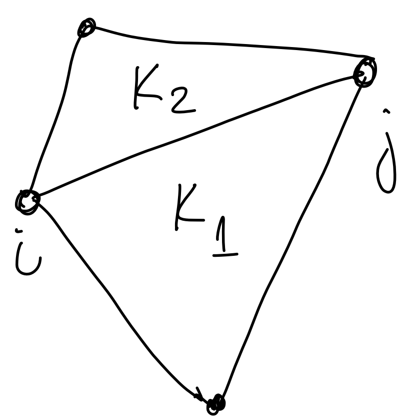
$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}$$



$|J| = 2A_{123} \leftarrow$ area of triangle

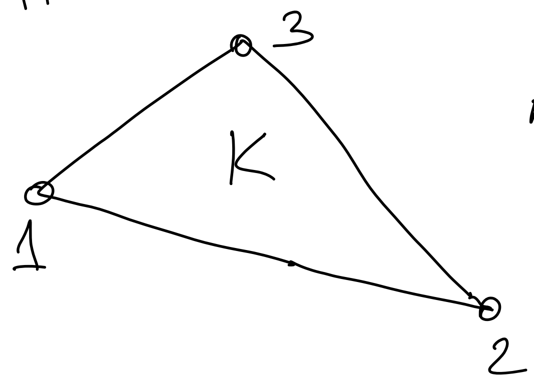
Recall that for Laplacian we need $\int \nabla\varphi_i \cdot \nabla\varphi_j \, dx$.

But the supports of φ_i and φ_j only overlap if nodes i and j are neighbors, and therefore



we get a nonzero contribution to the stiffness matrix from at most two triangles in 2D.

We therefore focus on a triangle K at a time, and assemble the stiffness matrix from triangle stiffness matrices



$$A^K_{ij} = \int_K \nabla\varphi_i \cdot \nabla\varphi_j \, dx$$

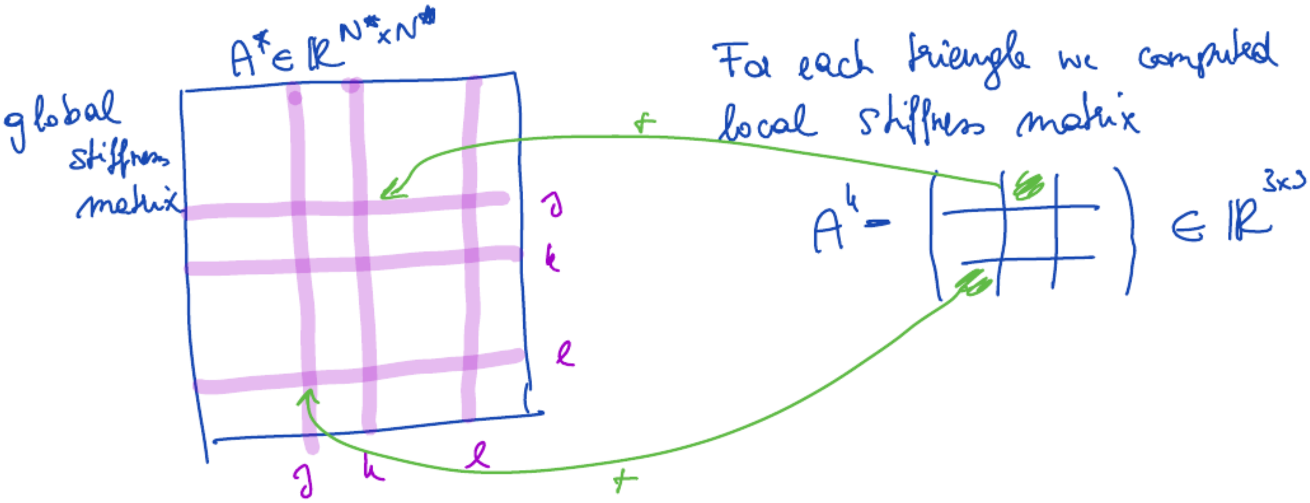
(3x3 matrix)

$$A^K = \frac{1}{4A_{123}} \begin{bmatrix} |r_2-r_3|^2 & (r_2-r_3)\cdot(r_3-r_1) & (r_2-r_3)\cdot(r_1-r_2) \\ & |r_3-r_1|^2 & (r_3-r_1)\cdot(r_1-r_2) \\ \text{Symmetric} & & |r_1-r_2|^2 \end{bmatrix}$$

local stiffness matrix, corresponding to triangle K

$$A = \sum_{K=1}^{M} A^K$$

Matrix assembly summary:

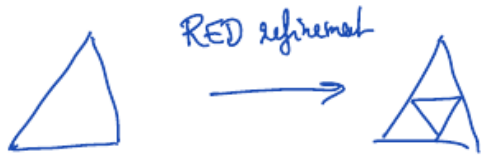


For each triangle we computed local stiffness matrix

All you need is a loop over triangles and an $M \times 3$ $(3 \times M)$ matrix mapping local DOFs to global DOFs

Refining grid based on a posteriori error estimate (from G. Stadler):

<u>In 2D:</u> How to refine a triangle



Split into 4 triangles, that are shape-regular.
Problem: What to do with neighboring element?

Split into 2 triangles. Problem: iterative GREEN refinement can result in poor shape regularity:

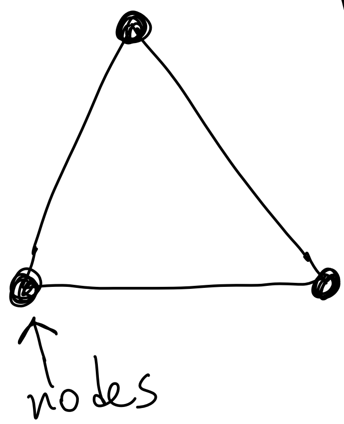
Splits one triangle into three, helps with shape regularity.

In practice, combine them:

24+1/2

Recall from interpolation lecture notes different elements & nodes in 2D:

① Linear triangles:



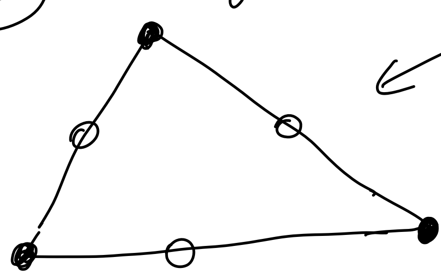
V_h = space of piecewise linear functions over triangle

Basis functions are tent functions

$U_i \equiv u(x_i)$

Functions in discrete space are continuous across edges, i.e, they are continuous on Ω.

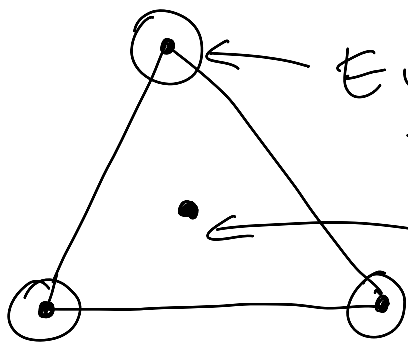
② Lagrange quadratic triangle



6 nodes, and now functions have continuous tangential derivatives along edges as well

③ Hermite triangle

$u(x)$ is now cubic on each element (dimension $= 10$ with basis $\{1, x, y, xy, x^2, y^2, x^3, y^3, x^2y, y^2x\}$)

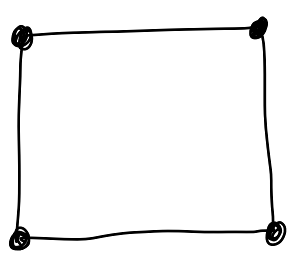


Evaluate function and its gradient $= 3$ dof's

Evaluate function only (1 dof)

$= 3 \times 3 + 1 = 10$ dofs total

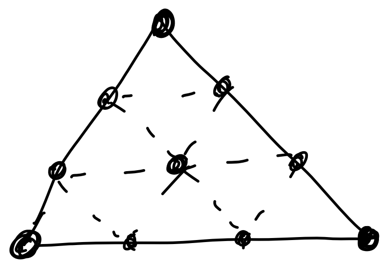
④ Bilinear rectangle



$P = \text{span}\{1, x, y, xy\}$ (4 dofs)

In general use tensor product of polynomials in x and in y; very simple but not all domains can be meshed with quadrilaterals.

5 Lagrange cubic triangle



Note that the global interpolant is still only C^0 since normal derivatives to an edge need not match

Sadly even Hermite triangles are not C^1! For higher order equations like biharmonic

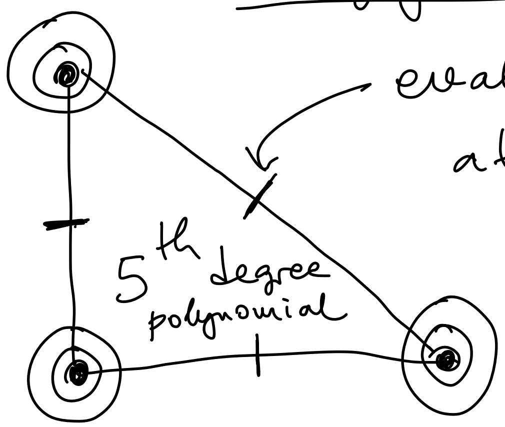
$$\nabla^4 u = f \quad , \quad \nabla u \cdot n = 0 \text{ on } \partial\Omega$$
$$u = 0 \text{ on } \partial\Omega$$

the suitable space is H_0^2.

Terminology: If $V_h \subset V$ (typical FEM), the FE approximation is conforming (otherwise non-conforming)

What element gives a conforming approximation to biharmonic eq.?

The Argyris triangle



evaluate normal derivative at midpoint

evaluate u, ∇u and $\nabla\nabla u$ = 6 dofs

(interpolation error in $H^2(\Omega)$ is $\sim h^4 \|u\|_{H^6}$)

$$\# \text{DOFs} = 3 \times 6 + 3 = 21$$

The number of DOFs grows rapidly as one increases the order, and FEM methods can be expensive especially for vector equations

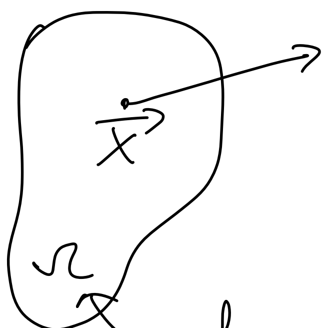
Another, probably better, alternative is to introduce a new variable

$$\begin{cases} \nabla^2 u = w \\ \nabla^2 w = f \end{cases} \rightarrow \text{mixed formulation}$$

Later we will mention <u>Discontinuous Galerkin (DEG)</u> as an alternative that avoids increasing the number of global dofs

Linear Elasticity in 2D

Equations are similar in structure to fluids but variable is displacement field $u(x) \in \mathbb{R}^2$ (not velocity)



$\vec{x} + \vec{u}(x)$ new position of point $\vec{x}$

reference configuration

Strain tensor (similar to strain rate for fluids)

$$\overleftrightarrow{e}(\vec{u}) = \frac{1}{2}\left(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^T\right) \in \mathbb{R}^{2\times 2}$$

Linear elasticity (small deformation)

Stress tensor

$$\overleftrightarrow{\sigma} = \overset{\langle\langle\longleftrightarrow\rangle\rangle}{L}\,\overleftrightarrow{e}$$

$$\sigma_{ij} = L_{ijkl}\, e_{kl}$$

implied summation

Isotropic material <u>must</u> have

$$L_{ijkh} = \lambda\, \delta_{ij}\delta_{kh} + \mu\left(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\right)$$

$$\begin{cases} \mu > 0 \\ \lambda + 2\mu > 0 \end{cases}$$

Lamé parameters (like viscosity, property of solid)

$$\Rightarrow \overleftrightarrow{\sigma} = 2\mu\, \overleftrightarrow{e} + \lambda\, \mathrm{Tr}(\overleftrightarrow{e})\, \overleftrightarrow{I}$$

trace

Strong form of PDE

$$\vec{\nabla} \cdot \overleftrightarrow{\sigma} = \vec{f} \leftarrow \text{body force (applied force)}$$

BCs are just like for Navier-Stokes: Specify one BC for <u>normal</u> direction (either $\vec{u}$ or $\overleftrightarrow{\sigma} \cdot \vec{n}$) and one for tangential (either $\vec{u}$ or $\overleftrightarrow{\sigma} \cdot \vec{\tau}$)

For essential BC (Dirichlet)

$\vec{u}(\partial\Omega) = \vec{0}$ we have an

<u>Energy formulation</u>:

$$\min_{u \in (H_0^1(\Omega))^2} \int_\Omega \left(\frac{1}{2} \sigma : e - f \cdot u \right) dx$$

$\sigma_{ij} e_{ij}$ (double contraction)

Weak formulation

$$\begin{cases} \nabla \cdot \sigma = f & \text{in } \Omega \\ \sigma \cdot n = g & \text{on } \partial\Omega_2 \\ u = 0 & \text{on } \partial\Omega_1 \end{cases}$$

Find $u \in \left(\bar{H}^1_0\right)^2$ s.t.

$$a(u, v) = l(v) \quad \forall v \in \left(\bar{H}^1_0\right)^2$$

$$u(\partial\Omega_1) = v(\partial\Omega_1) = 0$$

Where as before

$$a(u, v) = \frac{1}{2} \int_\Omega e(v) : L\, e(u)\, dx$$

$$l(v) = \int_\Omega f \cdot v\, dx + \int_{\partial\Omega_2} g \cdot v\, ds$$

Using a triangulation with linear hat basis functions

$$\{\eta_1, \dots, \eta_{2N}\} = \{ \varphi_1 e_1, \varphi_1 e_2, \varphi_2 e_1, \varphi_2 e_2, \dots \}$$

$$\Rightarrow \quad A U = F$$

$$U = \{ U_1^x, U_1^y, \dots, U_N^x, U_N^y \}$$

$$\equiv \{ U_1, U_2, \dots, U_{2N} \}$$

$$A_{kl} = \int_\Omega e(\eta_k) : L e(\eta_l) \, dx$$

$$F_k = \int_\Omega f \eta_k \, dx + \int_{\partial\Omega_2} g \cdot \eta_k \, ds$$

Explicit formulas can be obtained for linear triangles for the $(3\cdot 2)^2 = 6\times 6$ local / triangle stiffness matrix

Introduce

$$\gamma(u) = \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ 2e_{12}(u) \end{pmatrix}$$

$$\Rightarrow \quad e(v) : L e(u) = \gamma^T C \gamma \quad \text{where}$$

$$C = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

On a triangle

$$\gamma(u_h) = \underbrace{\begin{bmatrix} \varphi_{1x} & 0 & \varphi_{2x} & 0 & \varphi_{3x} & 0 \\ 0 & \varphi_{1y} & 0 & \varphi_{2y} & 0 & \varphi_{3y} \\ \varphi_{1y} & \varphi_{1x} & \varphi_{2y} & \varphi_{2x} & \varphi_{3y} & \varphi_{3x} \end{bmatrix}}_{R} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_6 \end{bmatrix}$$

Local DOFs on triangle

$$A^K = |K| \, R^T C R \quad \text{(clearly SPD)}$$

A more tricky example is

Stokes flow

$$\begin{cases} \nabla p = -\nabla \cdot \sigma + f \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } \Omega$$

$\sigma = -\eta\,(\nabla u + \nabla u^T)$, $\eta = \text{const}$

Now $\vec{u}$ is velocity

What we say here also applies to Navier-Stokes eqs.

Energy formulation:

$$\begin{cases} \min\limits_{v \in V} \dfrac{\eta}{2} \displaystyle\int_\Omega \nabla v : \nabla v \, dx + \int_\Omega f \cdot v \, dx \\ \text{s.t. } \nabla \cdot v = 0 \end{cases}$$

$\leftarrow p$ is Lagrange multiplier for constraint

Weak form involves different spaces for v and p : mixed FE.

Find $u \in (H_0^1(\Omega))^d$, $p \in L^2(\Omega)$

up to a constant

s.t.

$$\begin{cases} a(u,v) + b(v,p) = F(v) \\ \qquad\qquad b(u,q) = 0 \end{cases}$$

mixed bilinear form

$\forall (v,q) \in V \times Q$

saddle-point system

$$\begin{pmatrix} A & B^T \\ B & \emptyset \end{pmatrix} \begin{pmatrix} U \\ \underline{P} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Where $BU = 0$ defines the kernel space K (discretely divergence-free velocity fields)

$$a(u, v) = \eta \int_{\Omega} (\nabla u + \nabla^T u) : \nabla v \, dx$$

$$b(v, p) = \int_{\Omega} p \, (\nabla \cdot v) \, dx$$

Saddle-point system

$$v \cdot \Big[Au + B^T p = f$$

$$\Rightarrow v^T A u + (\underbrace{\cancel{Bv}}_{\text{zero}})^T p = v^T f$$

$$\Rightarrow \begin{cases} v^T A u = v^T f \\ \forall v \in K \\ (\text{variational problem on } K) \end{cases}$$

A key feature of Stokes flow is that the saddle-point system must be solveable & well-conditioned as $h \to 0$

Mathematically, this is expressed as the inf-sup condition (also called LBB = Ladyszenskaya, Brezzi, Babuskha):

$$\inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \gamma > 0$$

This condition means that pressure space Q_h cannot be "too large", since otherwise there will be some $q_h \in Q_h$ (a spurious or "parasitic mode") that will make the sup be zero.

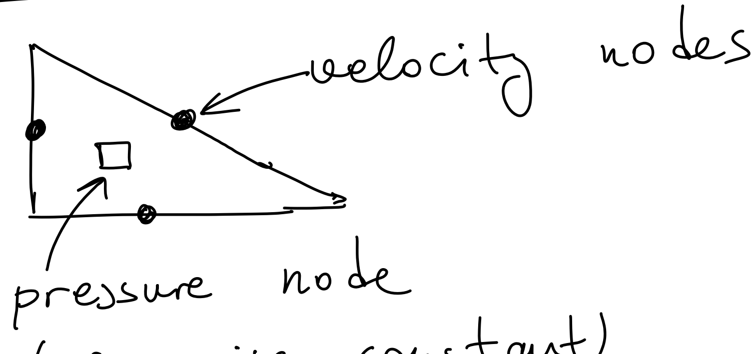
V_h and Q_h must be chosen together <u>not</u> independently

Using linear triangles for both pressure & velocity is **NOT** inf-sup stable and does not converge for Stokes.

Heuristically, the polynomial degree for pressure should be one **lower** than velocity.

Examples of stable elements:

Crouzeix – Raviart element

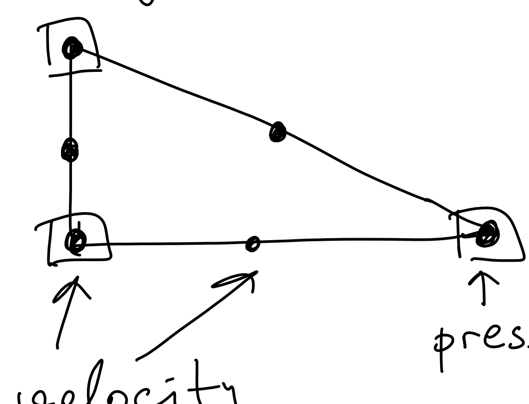


This is a non-conforming element since velocity is not continuous accross edges so $V_h \not\subset H^1$

It is in some sense the equivalent of the MAC or Staggered grid for triangles (but first order for velocity)

A more standard stable element is the

<u>Taylor–Hood</u> element



Standard quadratic Lagrange for velocity, linear for pressure

Already quite a bit more expensive than MAC!

Also stable is V_h order $k \geq 2$ polynomial for $\square$ cells, pressure degree $k-2$ discontinuous