

Implicit - Explicit (IMEX)

temporal integrators

A. DONEY, Fall 2021

Here are some standard 2nd order IMEX schemes for solving ODEs of the form:

$$u'(t) = \underbrace{f(u, t)}_{\substack{\text{not stiff} \\ \text{(explicit)} \\ \text{advection}}} + \underbrace{g(u, t)}_{\substack{\text{stiff} \\ \text{(implicit)} \\ \text{diffusion}}}$$

Often $g(u, t) = Lu$ is linear to avoid non-linear equations in implicit solve

Or $g(u, t) = L(t)u + g^{BC}(t)$ includes inhomogeneous BCs for diffusive terms (1)

Multistep AB2 + CN

Adams - Bashforth + Crank - Nicolson
(implicit midpoint)

See Section 11.5 in LeVeque

$$U^{n+1} = U^n + \frac{\tau}{2} \left(3f(U^n, t^n) - f(U^{n-1}, t^{n-1}) \right)$$

not
L-stable

$$+ \frac{\tau}{2} \left(g(U^n, t^n) + \underset{\substack{\uparrow \\ \text{implicit}}}{g(U^{n+1}, t^{n+1})} \right)$$

(11.26) in LeVeque

Backwards Differentiation

Semi-implicit BDF = SBDF2

$$U^{n+1} = \frac{4}{3} U^n - \frac{1}{3} U^{n-1} + \frac{2}{3} \tau g(U^{n+1}, t^{n+1})$$
$$+ \frac{2}{3} \tau \left[2f(U^n, t^n) - f(U^{n-1}, t^{n-1}) \right]$$

BDF2 is L-stable (8.3.2 in LeVeque) ②
(good for diffusion)

Runge-Kutta IMEX L-stable
RK2-RK2

$$U^{n+1/2,*} = U^n + \bar{\tau} \left(\left(1 + \frac{\sqrt{2}}{2}\right) g \left(U^{n+1/2,*}, t^n + \frac{\bar{\tau}}{2} \right) - \left(\frac{1 + \sqrt{2}}{2} \right) g \left(U^n, t^n \right) \right) + \frac{\bar{\tau}}{2} f \left(U^n, t^n \right)$$

(Predictor to midpoint)

$$U^{n+1} = U^n + \left(\left(1 + \frac{\sqrt{2}}{2}\right) g \left(U^{n+1}, t^{n+1} \right) - (1 + \sqrt{2}) g \left(U^{n+1/2,*}, t^n + \frac{\bar{\tau}}{2} \right) + \left(1 + \frac{\sqrt{2}}{2}\right) g \left(U^n, t^n \right) \right) + \bar{\tau} f \left(U^{n+1/2,*}, t^n + \frac{\bar{\tau}}{2} \right) \textcircled{3}$$

explicit midpoint

One can use explicit trapezoidal for $f(u)$ instead of midpoint, which is recommended for advection,

see paper in 2005 by Pareschi & Russo (see webpage).

For explicit RK3, the scheme recommended for advection is:

$$\left\{ \begin{array}{l} u^* = u^n + \bar{\tau} f^n \quad (\text{Euler step}) \\ u^{**} = \frac{3}{4} u^n + \frac{1}{4} [u^* + \bar{\tau} f^*] \\ \quad \underbrace{\hspace{10em}}_{\text{convex combination of Euler steps}} \\ u^{n+1} = \frac{1}{3} u^n + \frac{2}{3} [u^{**} + \bar{\tau} f^{**}] \\ \quad \underbrace{\hspace{10em}}_{\text{Third Euler step}} \end{array} \right.$$

It is possible to combine this explicit RK3 with the implicit RK2 from page 3 to obtain a great IMEX RK3-RK2 scheme for advection-diffusion MOL schemes. Here are the relevant pieces from the Pareschi + Russo 2005 paper:

Time-dependent inhomogeneous diffusion BCs go here

2. IMEX RUNGE-KUTTA SCHEMES

An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the source terms and an explicit one to the nonstiff term. When applied to system (1) it takes the form

$$\partial_t U = F(U, t) + R(U, t)$$

$$U^{(i)} = U^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} F(U^{(j)}) + \Delta t \sum_{j=1}^v a_{ij} R(U^{(j)}), \quad (3)$$

$$U^{n+1} = U^n - \Delta t \sum_{i=1}^v \tilde{w}_i F(U^{(i)}) + \Delta t \sum_{i=1}^v w_i R(U^{(i)}). \quad (4)$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (a_{ij})$ are $v \times v$ matrices such that the resulting scheme is explicit in F , and implicit in R . An IMEX Runge-Kutta scheme is characterized by these two matrices and the coefficient vectors $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_v)^T$, $w = (w_1, \dots, w_v)^T$.

The IMEX Runge–Kutta schemes can be represented by a double *tableau* in the usual Butcher notation,

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{w}^T \end{array} \quad \begin{array}{c|c} c & A \\ \hline & w^T \end{array},$$

where the coefficients \tilde{c} and c used for the treatment of non autonomous systems, are given by the usual relation

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, \quad c_i = \sum_{j=1}^i a_{ij}. \quad (5)$$

The use of a DIRK scheme for R is a sufficient condition to guarantee that F is always evaluated explicitly.

Table II. Tableau for the Explicit (left) Implicit (right) IMEX-SSP2(2,2,2) L-Stable Scheme RK2-RK2

0	0	0	γ	γ	0
1	1	0	$1-\gamma$	$1-2\gamma$	γ
	$1/2$	$1/2$		$1/2$	$1/2$

Use instead of scheme on page 3

$$\gamma = 1 + \frac{1}{\sqrt{2}}$$

I prefer \oplus sign

Table V. Tableau for the Explicit (left) Implicit (right) IMEX-SSP3(3,3,2) L-Stable scheme RK3-RK2

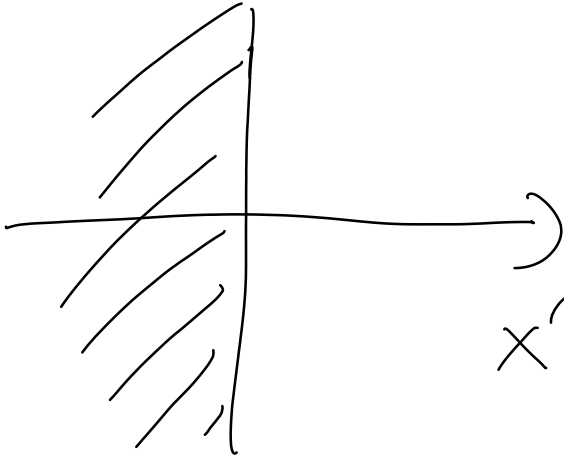
0	0	0	0	γ	γ	0	0
1	1	0	0	$1-\gamma$	$1-2\gamma$	γ	0
$1/2$	$1/4$	$1/4$	0	$1/2$	$1/2-\gamma$	0	γ
	$1/6$	$1/6$	$2/3$		$1/6$	$1/6$	$2/3$

$$\gamma = 1 + \frac{1}{\sqrt{2}}$$

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Aside L-stable scheme

(A-stable)



$$x'(t) = -\lambda x(t)$$

λ is large

$$x^n = \left(\frac{1 + \lambda \Delta t / 2}{1 - \lambda \Delta t / 2} \right) x^0$$

$$x^n = e^{-\lambda n \Delta t} x^0$$

decays rapidly

$$\lambda \Delta t \gg 1 \rightarrow x^n \rightarrow (-1)^n x^0$$

does

L-stable :

Stability $\left| R(\lambda \Delta t = z) \right| \leq 1$

L-stable = A-stable +

$$R(z \rightarrow \infty) \rightarrow 0$$

Backward Euler is

L-stable

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An L-stable scheme applied  
to a linear parabolic  
PDE with  $\Delta t \rightarrow \infty$   
will converge to the  
steady-state solution (elliptic)



## Good scheme:

- Stability  $\nu = \frac{a\bar{\tau}}{h} < C = O(1)$
- Diffusion only  $\Rightarrow$  strictly dissipative (L-stable)
- 2<sup>nd</sup> order