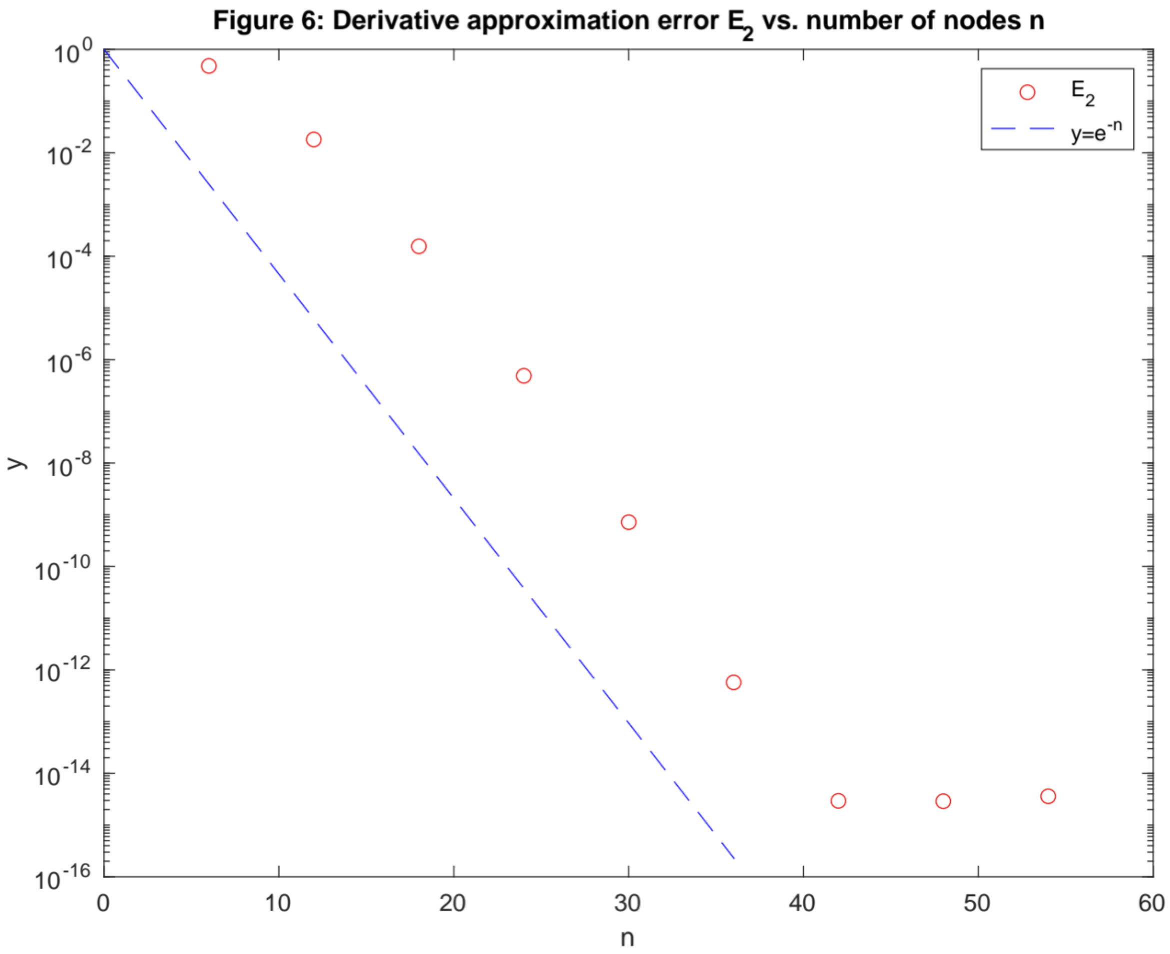
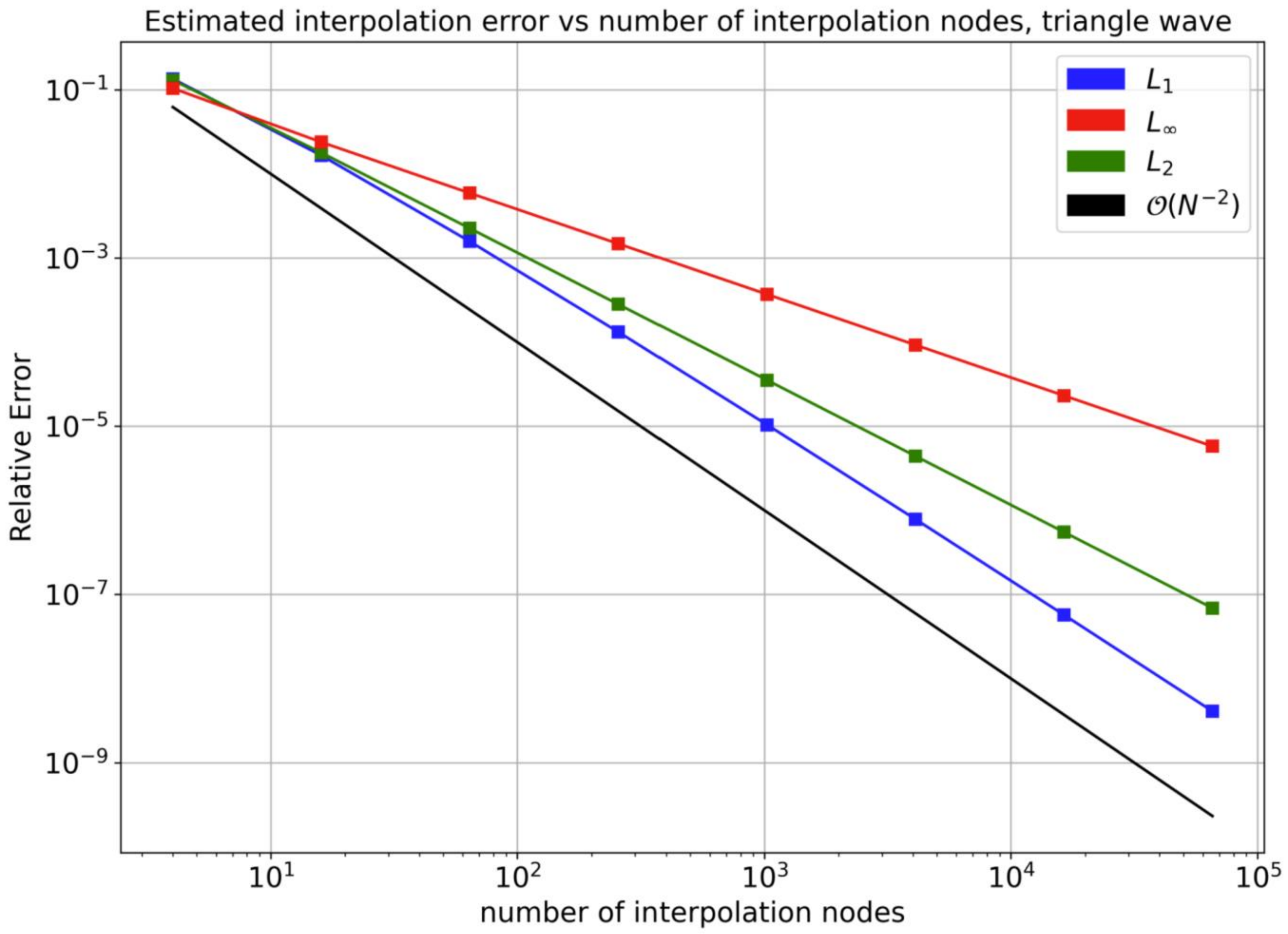
HW Reminders

- Submit a stand-alone **self-contained** PDF file
    - **Do not include code in your report** → codes should be submitted separately (.zip archive or as .txt files)
- Put your name in the filename of your report
- ***Read the instructions carefully***
- If you ever need to email me .py codes, attach them via **Google Drive**
- Make sure your figures are showing what you are trying to demonstrate

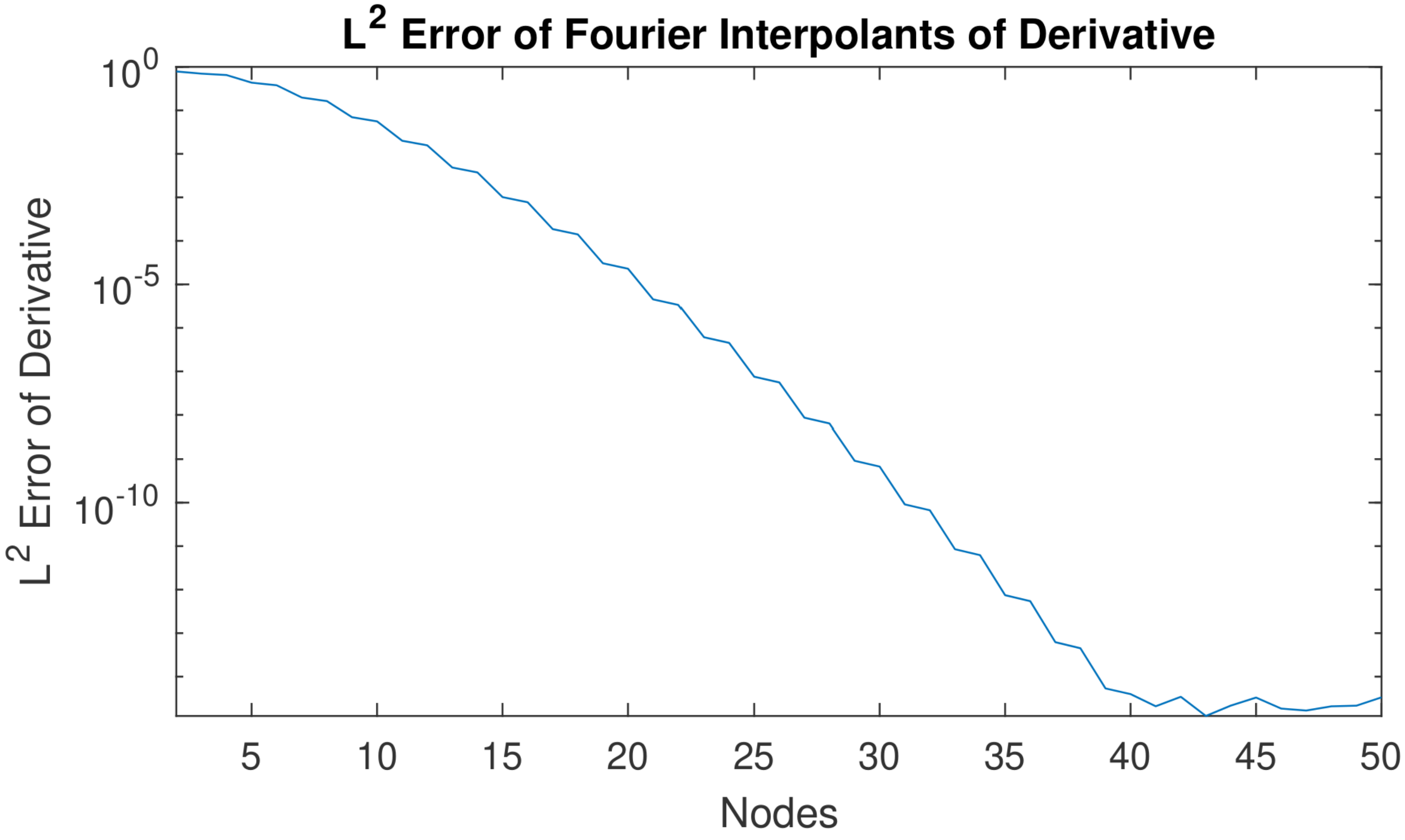
Log-Linear Plots



Log-Log Plots



A note on even sized grids



Working with even-sized grids

- Suppose you have a Fourier interpolant on an interval of size L using N nodes:

$$\tilde{\phi}(x) = \begin{cases} \hat{\phi}_0 + \sum_{0<k<\frac{N-1}{2}} \left(\hat{\phi}_k \exp(i\frac{2\pi}{L}kx) + \hat{\phi}_{N-1-k} \exp(-i\frac{2\pi}{L}kx) \right), & N \text{ odd}, \\ \hat{\phi}_0 + \sum_{0<k<\frac{N}{2}} \left(\hat{\phi}_k \exp(i\frac{2\pi}{L}kx) + \hat{\phi}_{N-k} \exp(-i\frac{2\pi}{L}kx) \right) + \hat{\phi}_{N/2} \cos(\frac{2\pi}{L}\frac{N}{2}x), & N \text{ even}, \end{cases}$$

- Suppose we want our interpolant on a finer grid $x_j = \frac{Lj}{M}, \quad M \gg N$

$$\tilde{\phi}(x_j) = \begin{cases} \hat{\phi}_0 + \sum_{0<k<\frac{N-1}{2}} \left(\hat{\phi}_k \exp(i\frac{2\pi j}{M}k) + \hat{\phi}_{N-1-k} \exp(-i\frac{2\pi j}{M}k) \right), & N \text{ odd}, \\ \hat{\phi}_0 + \sum_{0<k<\frac{N}{2}} \left(\hat{\phi}_k \exp(i\frac{2\pi j}{M}k) + \hat{\phi}_{N-k} \exp(-i\frac{2\pi j}{M}k) \right) + \hat{\phi}_{N/2} \cos(\frac{2\pi j}{M}\frac{N}{2}), & N \text{ even}, \end{cases}$$

The special $N/2$ mode for even N

$$\hat{\phi}_{N/2}\cos(\frac{2\pi j}{M}\frac{N}{2}x) = \frac{\hat{\phi}_{N/2}}{2}\exp(i\frac{2\pi j}{M}\frac{N}{2}) + \frac{\hat{\phi}_{N/2}}{2}\exp(-i\frac{2\pi j}{M}\frac{N}{2}),$$

- I.e. separate the $\hat{\phi}_{\frac{N}{2}}$ part into two halves → other half for the matched mode

The interpft routine:

- If N is even,

1. Use `fft` on the N interpolation points to obtain Fourier coefficients $\hat{\phi}_0, \cdots, \hat{\phi}_{N-1}$;
2. Apply oversampling to the Fourier coefficients
   $(\hat{c}_k)_{k=1}^{N} = (\hat{\phi}_0, \hat{\phi}_1, \cdots, \hat{\phi}_{N/2-1}, \frac{\hat{\phi}_{N/2}}{2}, 0, \cdots, 0, \frac{\hat{\phi}_{N/2}}{2}, \hat{\phi}_{N/2+1}, \cdots, \hat{\phi}_{N-1})$;
3. Apply `ifft` to $(\hat{c}_k)_{k=1}^{N}$, obtain the interpolation estimated on fine grids.

Spectral Differentiation

- What about for differentiation? Suppose:

$$\phi(x) = \hat{f}_0 + \sum_{0<k<\frac{n}{2}} \left(\hat{f}_k e^{ikx} + \hat{f}_{n-k} e^{-ikx} \right) + \hat{f}_{n/2} \cos(\frac{nx}{2}),$$

- Differentiating:

$$\begin{aligned}
\phi'(x) &= \sum_{0<k<\frac{n}{2}} \left(\hat{f}_k e^{ikx} \cdot ik + \hat{f}_{n-k} e^{-ikx} \cdot (-ik) \right) - \hat{f}_{n/2} \sin(\frac{nx}{2}) \cdot \frac{n}{2} \\
&= \sum_{0<k<\frac{n}{2}} \left(\hat{f}_k e^{ikx} \cdot ik + \hat{f}_{n-k} e^{-ikx} \cdot (-ik) \right) + \hat{f}_{n/2} \frac{1}{2} \left(e^{i\frac{n}{2}x} \cdot i\frac{n}{2} + e^{i(-\frac{n}{2})x} \cdot (-i\frac{n}{2}) \right),
\end{aligned}$$

The differentiation interpft routine:

1. Use `fft` on the n interpolation points to obtain Fourier coefficients $\hat{f}_0, \cdots, \hat{f}_{n-1}$;
2. Multiply the Fourier coefficients with the corresponding frequency (we do **not** zero out the special mode, but we would do this if $N = n$!)

$$\hat{\phi}'_k = \hat{f}_k \left(\frac{2\pi}{L}\right) \begin{cases} ik, & k \leq n/2, \\ -i(n-k), & k > n/2; \end{cases}$$

3. Apply oversampling to N Fourier coefficients
   $(\hat{c}_k)_{k=1}^{N} = (\hat{\phi}'_0, \hat{\phi}'_1, \cdots, \hat{\phi}'_{n/2-1}, \frac{\hat{\phi}'_{n/2}}{2}, 0, \cdots, 0, -\frac{\hat{\phi}'_{n/2}}{2}, \hat{\phi}'_{n/2+1}, \cdots, \hat{\phi}'_{n-1});$
4. Apply `ifft` to $(\hat{c}_k)_{k=1}^{N}$, obtain the derivative of interpolation estimated on fine grids.

The KdV Equation

- In Fourier space, we write the RHS of the KdV equation as

$$ik^3 \boxdot \hat{\boldsymbol{\phi}} - 3i\boldsymbol{k} \boxdot \boldsymbol{\mathcal{F}}\left(\left(\boldsymbol{\mathcal{F}}^{-1}\hat{\boldsymbol{\phi}}\right)^{\boxed{2}}\right),$$

- Dealing with the squared term:

1. Zero-pad $\hat{\phi}$ to N', ifft back to real space (with $N' \geq 3N/2$ for **anti-aliasing**)
2. Element-wise multiplication, fft back to fourier space
3. **Truncate back to N**

The KdV Equation: the routine

1. Compute the ϕ^2 term
2. Multiply the Fourier coefficients with the corresponding frequency to obtain $\widehat{\mathcal{K}[\phi(\cdot)]}$ (note that again we do **not** zero out the $N/2$ mode unless $M = N$)

$$F_k = \begin{cases} i(\frac{2\pi}{L}k)^3\hat{\phi}_k - 3i\frac{2\pi}{L}k\hat{\phi^2}_k, & k \le N/2, \\ i(-\frac{2\pi}{L}(N-k))^3\hat{\phi}_k - 3i\frac{2\pi}{L}(N-k)\hat{w}_k, & k > N/2, \end{cases}$$

3. Apply oversampling to M points:
   $(\hat{c}_k)_{k=1}^N = (F_0, F_1, \cdots, F_{N/2-1}, \frac{F_{N/2}}{2}, \cdots, -\frac{F_{N/2}}{2}, F_{N/2+1}, \cdots, F_{N-1});$
4. Apply `ifft` to $(\hat{c}_k)_{k=1}^N$, obtain the approximation $\mathcal{K}[\phi(\cdot, t)]$ estimated on fine grids.