

# Quick introduction to PDE, III

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## 1 Introduction

The first two sessions were largely mathematical rather than physical. But our interest in PDE may come (depending on the person) from their use in modeling physical systems. Section II (last week) gave an example of the derivation of a PDE as a continuum limit. Many other PDEs are derived in this way. This session will focus on another kind of reasoning used to derive a PDE, the use of local *conservation laws* and *constitutive relations* to find *fluxes* (also called *currents*) of the locally conserved quantities. These flux formulations of PDEs are often used to design numerical solution algorithms.

This session also describes the behavior of solutions of some PDEs through explicit solutions of simplified PDEs (linearized, constant coefficient, one dimensional, etc.). Solutions of “real” problems (without analytical solution) have qualitative behavior seen in these model problems. Therefore, the model problems are a guide to solution strategies for real problems.

## 2 Diffusion fluxes and equations

### Notation

In this session,  $x$  will be a one component position variable, often called the “space” variable. We will do PDEs only in one dimension. As before  $t$  is time and a dot over a quantity is the time derivative of that quantity. Densities will be called  $\rho$  or  $u$  or something like that. These will be functions of  $x$  and  $t$  and will satisfy PDEs. The flux of a quantity  $q$  will be called  $F_q$ . The amount of quantity  $q$  in an interval  $a \leq x \leq b$  will be

$$M_q(t, a, b) = \int_a^b q(x, t) dx . \quad (1)$$

$M$  is for “mass”, which is what  $M_\rho$  represents if  $\rho$  represents mass density, but if  $q$  is not  $\rho$  then  $M_q$  could be the total momentum or the total energy, etc.

### Diffusion of a substance

Imagine a small drop of color (food color or ink) at the bottom of a glass of still water. Over time, the drop will slowly spread throughout the water until the color becomes uniform. This is not easy to do “at home”, which may explain why the Youtube videos I found of food color diffusion in water all involved more complicated processes of gravity (buoyancy) and motion in the water (advection). Here is a description of simple diffusion of a substance such

as food color in a one dimensional “glass” of “water”. You might think of this 1D (one dimensional) water as filling a long thin straight glass tube. We let  $\rho(x, t)$  be the density<sup>1</sup> of color at location  $x$  at time  $t$ . Then (1) gives  $M(t, a, b)$  as the amount of color between  $a$  and  $b$  at time  $t$ .

We derive a PDE for  $\rho(x, t)$  by first finding a model for  $\dot{M}$ . The main idea is that color enters or leaves the interval  $[a, b]$  only by crossing one of the boundary points  $a$  or  $b$ . We need a model for these rates. These rates are *fluxes* or *currents*,  $F(x, t)$ . The flux  $F(a, t)$  represents the net rate at which color crosses from  $x < a$  to  $x > a$  per unit time. A positive flux at  $a$ ,  $F(a, t) > 0$ , encourages  $M(t, a, b)$  to increase while positive flux at  $b$  encourages  $M$  to decrease:

$$\dot{M}(t, a, b) = -F(b, t) + F(a, t) . \quad (2)$$

This is a *local conservation law*, which expresses the model assumption that color can move but it is not created or destroyed. Moreover, color moves without “jumping” over boundaries. It must cross  $x = a$  to go from  $x < a$  to  $x > a$ .

The conservation relation (2), by itself, does not describe the dynamics of  $M$  or  $\rho$  completely. We also need a *constitutive relation* that specifies the flux  $F$  in terms of the color distribution. This is more detailed physical modeling, beyond the generic local conservation assumption (2). *Fick’s law* is an “entry level” constitutive model. It says that color “flows” from regions of high color density to regions of lower density at a rate that is proportional to the color density gradient. We write  $D$  for the constant of proportionality, and assume it’s positive. The flux, according to Fick’s law, is

$$F(x, t) = -D \partial_x \rho(x, t) . \quad (3)$$

The diffusion coefficient is a physical property of the water and the color material that is determined by microscopic physics. If the molecules of color are large, they diffuse slowly, which would be reflected in a small  $D$ . A small  $D$  means that it takes a large gradient  $\partial_x \rho$  to achieve a given flux  $F$ .

The conservation equation (2) and Fick constitutive law (3) are enough to give a PDE for the evolution of  $\rho$ . A first step is to write a local and differential form of the conservation relation (2). Both sides of (2) may be expressed as integrals, one side using (1) and the other side using basic calculus:

$$\frac{d}{dt} \int_a^b \rho(x, t) dx = \int_a^b \partial_t \rho(x, t) dx = - \int_a^b \partial_x F(x, t) dx .$$

If  $\rho$  and  $F$  are continuously differentiable, and if this holds for all  $t$  and arbitrarily small intervals  $[a, b]$ , then it must hold “pointwise” for every  $x$  and  $t$ :

$$\partial_t \rho(x, t) = -\partial_x F(x, t) . \quad (4)$$

The local (and differential) conservation relation (4) is equivalent to the global (and integral) form (2), provided that  $\rho$  and  $F$  are sufficiently differentiable.

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<sup>1</sup>Technically, this is the *volumetric* density, of the amount per unit volume, rather than the *specific* density, which is the amount per unit of water. Unit “volume” is unit length in a 1D problem.

Like global conservation, local conservation is a basic physical principle that does not depend on specific physical properties of the materials (constitutive relations).

The final PDE is found by combining the local conservation relation (4) with the constitutive law, which is Fick's law (3) in this case. The result is

$$\partial_t \rho(x, t) = D \partial_x^2 \rho(x, t) . \quad (5)$$

This is “the” heat equation, or “the” diffusion equation, depending on whether you're modeling heat flow (see below) or diffusion (as in the present diffusion of color dye).

The mathematics of the equation and qualitative properties of the solution of this PDE do not depend on what's being modeled. In particular, if there are boundaries at  $a$  and  $b$ , and if the boundary conditions cooperate (see below), there is a dissipation relation

$$\frac{d}{dt} \frac{1}{2} \int_a^b \rho(x, t)^2 = -D \int_a^b (\partial_x \rho(x, t))^2 dx . \quad (6)$$

This had a physical interpretation when we interpreted the unknown as modeling displacements of a chain of masses connected by springs. Here, it may not be clear how to interpret the square of the density on the left side, or whether to interpret the integrand on the right as the square of the flux. Depending on the boundary conditions, it may be possible to express the solution as a sum of modes.

### Boundary conditions

The most common boundary conditions for diffusion problems are *Dirichlet*<sup>2</sup> and *Neumann*.<sup>3</sup> Dirichlet boundary conditions are

$$\rho(a, t) = 0 , \quad \rho(b, t) = 0 . \quad (7)$$

These would apply in diffusion of color if there were something at the ends that absorbs color (maybe a form of chemical glue?). This can happen in diffusion problems when there is a chemical reaction that consumes the substance that's diffusing, where the reaction happens only at the boundary. The catalytic converter in a car is like this – a chemical you don't want released into the air reacts with a material on the wall, palladium is common.

Neumann boundary conditions are

$$\partial_x \rho(a, t) = 0 , \quad \partial_x \rho(b, t) = 0 . \quad (8)$$

They would apply for diffusion in a tube if the ends were closed. Closed ends means there is no flux in or out of the tube, which is  $F(a, t) = F(b, t) = 0$ .

<sup>2</sup>The traditional American pronunciation: starts with “dear”, continues with “ish” as in “wish” and ends with “sleigh”.

<sup>3</sup>The traditional American pronunciation starts with “oy” as in “boy”, so “Neu” sounds like the end of “annoy”. It ends with “on” as in the end of “coupon”.

Fick's law (3) relates the flux to the derivative. You can check using integration by parts as last week that the “energy identity” (6) applies with either Dirichlet or Neumann boundary conditions.

The solution is different for different boundary conditions. We saw last week that with Dirichlet boundary conditions,  $\rho(x, t) \rightarrow 0$  exponentially as  $t \rightarrow \infty$ . This is impossible with Neumann boundary conditions if the initial data  $\rho(x, 0)$  are positive, because the total amount of color (whatever is diffusing) cannot change if the flux is zero at the ends:

$$\frac{d}{dt} \int_a^b \rho(x, t) dx = -F(b, t) + F(a, t) = 0 .$$

### Fundamental solution

If the initial data is  $\rho(x, 0) = \delta(x - x_0)$ , then the solution is called the *fundamental solution*, and may be written  $G(x, x_0, t)$ . This is the solution to the diffusion problem if you start with no color anywhere except for a concentration at the point  $x_0$ . The fundamental solution, like any other solution, depends on the boundary conditions. Thus, there are Dirichlet and Neumann fundamental solutions. The simplest one is the “free space” fundamental solution, which we call  $G_0(x, x_0, t)$ .

The solution to any initial value problem can be expressed as an integral involving the fundamental solution. You can think of an initial distribution  $\rho(x, 0)$  as being small “drops” of stuff at each point  $x_0$ , with  $\rho(x_0, 0)$  being the size of the drop. Since the PDE is linear, the solution for all these drops (infinitely many infinitely small drops) is the sum of the solutions for the individual drops. That is

$$\rho(x, t) = \int_a^b G(x, x_0, t) \rho(x_0) dx_0 .$$

All this is a little vague, but it is supposed to explain “what’s going on”.

The *free space fundamental solution* for the diffusion equation (5) with  $x_0 = 0$  is

$$G_0(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} . \quad (9)$$

The term *free space* means there are no boundaries or boundary conditions. The function  $G_0$  is defined for all  $x$ . The term *fundamental solution* refers to the fact (explained below) that any solution may be expressed in terms of  $G_0$  through an integral. There are different fundamental solutions for different domains and boundary conditions, but the formulas for them are more complicated, and most of them are sums involving  $G_0$ . In this sense, the free space fundamental solution is the most fundamental solution. We first explain some properties of this formula, then we explain how they help understand solutions of the diffusion equation

• **It is a solution.** The time derivative is

$$\begin{aligned} \partial_t \left[ \frac{1}{\sqrt{4\pi D}} t^{-\frac{1}{2}} e^{-\frac{x^2}{4D}t^{-1}} \right] &= \frac{1}{\sqrt{4\pi D}} \left[ -\frac{1}{2} t^{-\frac{3}{2}} + t^{-\frac{1}{2}} \frac{x^2}{4D} t^{-2} \right] e^{-\frac{x^2}{4D}t^{-1}} \\ &= \frac{1}{\sqrt{4\pi D}} \frac{1}{2t^{\frac{3}{2}}} \left[ \frac{x^2}{2Dt} - 1 \right] e^{-\frac{x^2}{4D}t^{-1}} \end{aligned} \quad (10)$$

The space differentiation calculation is

$$\begin{aligned} \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} &\xrightarrow{\partial_x} \frac{1}{\sqrt{4\pi Dt}} \left[ -\frac{x}{2Dt} e^{-\frac{x^2}{4Dt}} \right] \\ &\xrightarrow{\partial_x} \frac{1}{\sqrt{4\pi Dt}} \left[ \frac{x^2}{4D^2t^2} - \frac{1}{2Dt} \right] e^{-\frac{x^2}{4Dt}} \\ &= \frac{1}{\sqrt{4\pi D}} \frac{1}{2t^{\frac{3}{2}}} \frac{1}{D} \left[ \frac{x^2}{2Dt} - 1 \right] e^{-\frac{x^2}{4Dt}} \end{aligned} \quad (11)$$

Compare the results (10) and (11), and you see that  $\partial_t G_0 = \partial_x^2 G_0$ . This shows that  $G_0$  satisfies the diffusion equation (5).

• **It integrates to 1.** The basic Gaussian integral formula is

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}.$$

Putting in a scaling  $y^2 = az^2$ , or  $z = \sqrt{a}y$  gives ( $\sqrt{a}$  goes “downstairs” because the integral is smaller when  $a$  is larger)

$$\int_{-\infty}^{\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}}.$$

Take  $a = \frac{1}{4Dt}$  and you get

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}} dx = \sqrt{4\pi Dt}.$$

Therefore the free space fundamental solution (9) satisfies

$$\int_{-\infty}^{\infty} G_0(x, t) dx = 1.$$

The total mass of  $G_0$  is 1.

• **It is localized.** The quadratic exponential  $e^{-y^2}$  converges to zero rapidly as  $|y| \rightarrow \infty$ . The factor  $\frac{1}{4Dt}$  in the exponent of (9) makes it converge to zero faster when  $t$  is smaller. “Clearly” (you can convince yourself in a few minutes) this implies that most of the unit mass of  $G_0$  is concentrated near  $x = 0$  when  $t$  is small. For example, the mass farther than  $\epsilon$  from  $x = 0$  disappears as  $t \rightarrow 0$ :

$$\int_{|x| > \epsilon} G_0(x, t) dx \rightarrow 0 \text{ as } t \rightarrow 0.$$

If you want the solution of the diffusion equation to be the fundamental solution, you have to take initial data that consists of a unit amount of stuff all concentrated at the point  $x = 0$ . This density “function” is called the *delta function* and it is written  $G_0(x, 0) = \delta(x)$ . The delta function, or the delta “distribution” is the most localized any distribution can be.

• **It is not localized.** The fundamental solution is the distribution that results from initial data putting a unit amount of stuff (color) at  $x = 0$  at time 0. Then you watch it diffuse. At any positive time  $t > 0$ , the density of stuff at  $x$ , which is  $G_0(x, t)$ , is positive. This is true no matter how far  $x$  is from the source point  $x = 0$  or how little time has passed. Diffusion allows stuff to move arbitrarily quickly. The localization comes not from  $G_0(x, t)$  being equal to zero, it isn't. It comes from  $G_0(x, t)$  being very small when  $x$  is large or  $t$  is small. If  $x$  is large or  $t$  is small, the approximation  $G_0(x, t) \approx 0$  may be accurate enough.

If  $\rho(x, t)$  satisfies the diffusion equation for all  $x$  and all  $t \geq 0$  with initial data  $\rho(x, 0)$  given, then the solution at later time may be expressed as an integral involving the fundamental solution

$$\rho(x, t) = \int_{-\infty}^{\infty} G_0(x - x_0, t) \rho(x_0, 0) dx_0 . \quad (12)$$

Informally, this means that whatever stuff (color) started at  $x_0$  at time  $t = 0$  diffuses according to the fundamental solution. The density at  $x$  at time  $t$  is the sum (integral) of the density at  $x$  at time  $t$  that came from  $x_0$  at  $t = 0$ . The points  $x_0$  close to  $x$  contribute a larger share. More formally, you can check that the fundamental solution representation formula (12) satisfies the diffusion equation (differentiate under the integral sign with respect to  $x$  and  $t$  and use the fact that  $G_0$  is a solution). You can check that  $\rho(x, t) \rightarrow \rho(x, 0)$  as  $t \downarrow 0$  because  $G_0$  is localized.

The fundamental solution representation formula provides a way to see some properties diffusion processes. One is *smoothing*. The solution  $\rho(x, t)$  is a differentiable function of  $x$  if  $t > 0$  even if the initial data  $\rho(x, 0)$  has discontinuities. As a mathematical argument for this, you can see that  $\partial_x$  applied to the left side results in  $\partial_x$  being applied to  $G_0$  on the right side, and  $G_0$  is a differentiable function of  $x$ . We saw the smoothing effect in a different way last week using the decay of Fourier series coefficients when  $t > 0$ . These approaches are complementary.

### 3 Advection

*Advection* and *convection* refer to material (color) or heat being *transported* by moving fluid (air or water or some other liquid or gas). A simple model of advection in 1D has an *advection velocity*,  $u(x)$ , that carries stuff with it. You can write a PDE for this using the *advective flux*

$$F(x, t) = u(x) \rho(x, t) . \quad (13)$$

This says that the rate at which stuff crossed a specific point  $x$  at time  $t$  is proportional to the density and to the advection velocity. The local conservation relation (4) then gives  $\partial_t \rho = -\partial_x(u\rho)$ . This is traditionally written with this with the “advection term” on the other side:

$$\partial_t \rho(x, t) + \partial_x [u(x)\rho(x, t)] = 0 . \quad (14)$$

This is “the” *advection equation*.

The advection equation with a constant advection velocity  $u(x) \equiv u$  has simple solutions that illustrate the solution of more complicated advection equations. With a constant  $u$ , it comes out of the differentiation, leaving

$$\partial_t \rho + u \partial_x \rho = 0 . \quad (15)$$

In the two dimensional time-space space, the left side represents directional derivative in the direction  $(1, u)$ , as in

$$[1 \cdot \partial_t + u \cdot \partial_x] \rho(x, t) = 0 .$$

This means that  $\rho$  is constant in the  $(1, u)$  direction. Lines in the  $(1, u)$  direction are *characteristic lines*. The characteristic line that starts at  $x_0$  when  $t = 0$  is given by  $x = x_0 + ut$ . The constant velocity advection equation implies that  $\rho$  is constant on these characteristic lines. Therefore, the value of the solution at a point  $(x, t)$  is determined by its value on the same line at  $t = 0$ . This leads to the formula

$$\rho(x, t) = \rho(x - ut, 0) . \quad (16)$$

You can check by direct differentiation that this characteristic solution formula (16) satisfies the constant velocity advection equation (15) and has the right initial conditions.

The characteristic line solution formula (16) implies that, for constant velocity advection, the density function  $\rho$  is carried (“advected”) by the velocity at speed  $v$  without changing shape. Unlike diffusion, advection does not smooth discontinuities. If  $\rho(x, 0)$  is discontinuous at  $x_0$ , then  $\rho(x, t)$  is discontinuous at  $x_0 + ut$ .

### Advection and diffusion together

Suppose the “stuff” is diffusing as it is being advected. You can model this by writing the flux as a sum of *advective* and *diffusive* fluxes. These are as given in (13) and (3):

$$F_{\text{tot}} = F_{\text{ad}} + F_{\text{diff}} = u\rho - D\partial_x \rho . \quad (17)$$

This leads to the *advection diffusion equation*. By tradition, the terms corresponding to advection and diffusion are on opposite sides of the equation;

$$\partial_t \rho(x, t) + \partial_x [u(x)\rho(x, t)] = D\partial_x^2 \rho(x, t) . \quad (18)$$

If the advection velocity is constant, there is a fundamental solution to the advection diffusion problem (18) that corresponds to the fundamental solution of the pure diffusion equation being advected with velocity  $u$ :

$$\rho(x, t) = G_0(x - ut - x_0, t) = \frac{1}{\sqrt{4Dt}} e^{-\frac{(x-ut-x_0)^2}{4Dt}}. \quad (19)$$

This what advection and diffusion produce if you start with a unit mass at  $x_0$ . The density field  $\rho$  “advects” with speed  $u$  while it diffuses with diffusion coefficient  $D$ . A peak in the initial density moves by advection while spreading by diffusion.

### Variable speed advection

The problem of “pure” advection (no diffusion) with variable speed has a semi-explicit solution in terms of characteristic curves. These curves add mathematical complexity, and there is a physical complexity related to compression or “rarefaction” (spreading) that comes from conservation together with a variable advection speed.

Qualitatively, suppose  $u(x) > 0$  when  $x < 0$  and  $u(x) < 0$  when  $x > 0$ . This means that the “wind” (advection velocity) is “blowing” to the right if  $x < 0$  and to the left when  $x > 0$ . That means the stuff with density  $\rho(x, t)$  is being concentrated near  $x = 0$ . This causes  $\rho$  to increase. The density,  $\rho$  must increase if the same amount of stuff is concentrated in smaller regions of space.

The math of characteristic curves gives formulas that describe this picture. We write  $\xi(t, x_0)$  for the *characteristic curve* that moves with speed  $u$  and starts at  $x_0$  at  $t = 0$ . In formulas,

$$\dot{\xi}(t, x_0) = u(\xi(t, x_0)), \quad \xi(0, x_0) = x_0.$$

At a point  $x, t$ , the *compression rate* is<sup>4</sup>

$$r(x) = -\partial_x u(x).$$

“Compression” corresponds to  $r > 0$  and anti-compression, or “rarefaction” corresponds to  $r < 0$ . If  $r > 0$ , the characteristic curve at  $x + \Delta x$  has speed  $u(x + \Delta x) \approx u(x) - r(x)\Delta x < u(x)$ . That is, the curve “in front” at  $x + \Delta x$  moves more slowly than the one at  $x$ . This makes the curves come closer together. Similarly,  $r < 0$  implies that curves are spreading apart.

Variable speed advection has the property that the total mass between two characteristic curves does not change with time. That corresponds to the stuff being “passively” advected by the velocity field  $u$ . Consider the characteristic curves  $\xi(t, x_0)$  and  $\xi(t, x_1)$  starting at  $x_0 < x_1$ . Characteristic curves cannot cross because their velocity is determined by their position. so the curve trying

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<sup>4</sup>It might be better to call this  $c(x)$  rather than  $r(x)$ , for “compression rate”. We use  $r$  because  $c(x)$  is often used for the advection velocity, which we call  $u(x)$ .



to cross from the left cannot go faster than the curve it's trying to cross.<sup>5</sup> The mass between these curves at time  $t$  is the integral of the density, as in (1):

$$M(t, x_0, x_1) = \int_{\xi(t, x_0)}^{\xi(t, x_1)} \rho(x, t) dx .$$

When you differentiate with respect to  $t$ , there are terms corresponding to the fact that the boundaries (endpoints) are moving. Increasing the lower boundary makes the integral decrease by an amount proportional to  $\rho$  at the lower boundary. Increasing the upper boundary makes the integral increase. The calculation uses this, together with the “conservation form” advection equation (18):

$$\begin{aligned} \frac{d}{dt} M(t, x_0, x_1) &= -\dot{\xi}(t, x_0) \rho(\xi(t, x_0), t) + \dot{\xi}(t, x_1) \rho(\xi(t, x_1), t) \\ &\quad + \int_{\xi(t, x_0)}^{\xi(t, x_1)} \partial_t \rho(x, t) dx \\ &= -u(\xi(t, x_0)) \rho(\xi(t, x_0), t) + u(\xi(t, x_1)) \rho(\xi(t, x_1), t) \\ &\quad + \int_{\xi(t, x_0)}^{\xi(t, x_1)} \partial_x [u(x) \rho(x, t)] dx \\ &= 0 . \end{aligned}$$

Since  $M$  is the integral of  $\rho$  over an interval between characteristics, and since  $M$  doesn't change, if the interval gets smaller then the integrand (the density) must get bigger.

### Nonlinear problems

The equations listed so far are linear but the problems solved on the computer are likely to be non-linear. Section 4 gives an example of a nonlinear PDE. But even without that complexity, an advection diffusion equation could be nonlinear if the diffusion coefficient depends on  $\rho$ . This might happen, for example, if  $\rho$  represents heat. Heat diffusion coefficients (usually called “heat conduction” coefficients) may be temperature dependent. Diffusion coefficients can depend on concentration, etc. Most phenomena present in linear problems also appear in nonlinear problems, but some things happen in nonlinear problems only.

## 4 Acoustics

*Acoustics* refers to propagation of sound. Sound can propagate in air, in water, and in solids. A PDE that models acoustics should explain how the speed of

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<sup>5</sup>Feel free to write a more mathematical version of this argument. For example, if curves would cross, then there would be a  $t_*$  with  $\xi(t_*, x_0) = \xi(t_*, x_1)$ . The uniqueness theorem for ODE solutions would imply that  $\xi(t, x_0) = \xi(t, x_1)$  for all  $t$ .

sound arises from material properties of the medium (air, water, ..) the sound is propagating in. It should explain why sound propagates at a finite speed (unlike diffusion).

You can find sound propagation in a model of a one dimensional motion of a compressible gas (air). A simple model<sup>6</sup> involves dynamical distributed parameters  $\rho(x, t)$ , the density of the gas, and  $v(x, t)$ , the velocity. The picture is that air at a point  $(x, t)$  has density  $\rho(x, t)$  and is moving with velocity  $v(x, t)$ . We rely on local conservation principles for mass and momentum. The mass between points  $a$  and  $b$  is (sorry for the terrible notation)

$$M_q(a, b, t) = \int_a^b \rho(x, t) dx .$$

The momentum density is  $\rho(x, t)v(x, t)$ . The amount of momentum between  $x$  and  $x + \Delta x$  is the mass in that interval, which is approximately  $\rho(x, t)\Delta x$ , and the velocity of that mass, which is  $v(x, t)$ . The momentum, being mass $\times$ velocity, is approximately  $\rho(x, t)v(x, t)\Delta x$ . You get the total momentum between two points by integrating the momentum density

$$M_p(a, b, t) = \int_a^b \rho(x, t) v(x, t) dx .$$

There are local fluxes corresponding to each locally conserved quantity, which will be the *mass flux*,  $F_q$ , and the *momentum flux*,  $F_p$ .

The mass flux is the same as it was for advection. The rate of stuff (gas) crossing a point is the product of the density and the velocity

$$F_q(x, t) = \rho(x, t)v(x, t) .$$

The momentum flux is more subtle. There are two ways momentum can cross a point, advection of momentum and pressure. The advection of momentum comes from the fact that gas crossing a point carries momentum with it. The flux of advected momentum is the momentum density multiplied by the velocity. The momentum density (as explained above) is  $\rho(x, t)v(x, t)$ , so the advection momentum flux is  $(\rho(x, t)v(x, t))v(x, t) = \rho(x, t)v^2(x, t)$ .

The “physics” of this *gas dynamics* model is the model of  $p(x, t)$ , the pressure at  $(x, t)$ . *Pressure* is a force that one part of a gas applies to the part next to it or to the ends of a tube or the boundary if it’s more than one dimensional. For example, in a balloon, the pressure of the gas inside pushing out on the rubber of the balloon is balanced by the force of the stretched rubber pushing in on the gas, and the lower pressure of the gas outside. Weather balloons expand as the rise because the outside pressure decreases. We saw that force is the rate of change of momentum. Therefore  $\frac{d}{dt}M_p(a, b, t)$  should have contribution from pressure at  $a$  and  $b$ . To get the signs right. A high pressure at  $a$  makes the momentum on the right “want to” increase and the pressure on the left makes

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<sup>6</sup>This model is wrong in its basic physics. However, “all models are wrong, some models are useful” – George Box.

it want to decrease. The flux corresponding to pressure is just  $p(x, t)$ . Thus, the total momentum flux is

$$F_p(x, t) = \rho(x, t)v^2(x, t) + p(x, t) .$$

The local conservation law in differential form for the mass is

$$\begin{aligned} \partial_t \rho(x, t) + \partial_x F_q(x, t) &= 0 \\ \partial_t \rho(x, t) + \partial_x [\rho(x, t)v(x, t)] &= 0 . \end{aligned}$$

The physics in this formula is only local conservation of mass and the model that a gas has a local density and velocity. The momentum density is  $\rho(x, t)v(x, t)$ , and the differential form of the local conservation of momentum is

$$\begin{aligned} \partial_t [\rho(x, t)v(x, t)] + \partial_x F_p(x, t) &= 0 \\ \partial_t [\rho(x, t)v(x, t)] + \partial_x [\rho(x, t)v^2(x, t) + p(x, t)] &= 0 . \end{aligned}$$

This is as far as we can go without making a more detailed model of the local properties of a gas.

The assumption of simple gas dynamics is that there is a *thermodynamic* relation so that the pressure at a point is a function of the density at that point. We write this (with some conflict of notation) as

$$p(x, t) = p(\rho(x, t)) .$$

The basic assumption is that the pressure is an increasing function of the density. This allows us to write

$$c^2(\rho) = \sqrt{\frac{dp}{d\rho}} > 0 . \tag{20}$$

The final equations of compressible gas dynamics are

$$\partial_t \rho(x, t) + \partial_x [\rho(x, t)v(x, t)] = 0 \tag{21}$$

$$\partial_t [\rho(x, t)v(x, t)] + \partial_x [\rho(x, t)v^2(x, t) + p(\rho(x, t))] = 0 . \tag{22}$$

The equations may be more clear if we leave out the  $(x, t)$  arguments

$$\begin{aligned} \partial_t \rho + \partial_x (\rho v) &= 0 \\ \partial_t (\rho v) + \partial_x (\rho v^2 + p(\rho)) &= 0 . \end{aligned}$$

These equations are nonlinear and we won't be able to analyze them much by formulas.

### Linear acoustics

*Linear acoustics* is a linear approximation to the nonlinear conservation law equations (21) and (22). For this, we assume a “small disturbance” about still air, where “still air” means  $\rho(x, t) = \rho_0 = \text{constant}$ , and  $v(x, t) = 0$ . Note

that constant  $\rho$  and  $v$  is a (trivial) solution to the dynamical equations. The small disturbance is  $\epsilon\tilde{\rho}$  and  $\epsilon\tilde{v}$ . The total density, background and disturbance is  $\rho(x, t) = \rho_0 + \tilde{\rho}(x, t)$ .

We find the linearized equations by substituting in the small disturbance assumption and approximating all the terms including terms of order  $\epsilon$  but dropping terms of order  $\epsilon^2$  or higher. For example, the flux in the mass conservation equation is

$$\rho v = (\rho_0 + \epsilon\tilde{\rho})\epsilon v = \rho_0 + \epsilon\rho_0\tilde{v} + O(\epsilon^2) .$$

This is also the momentum density (first term on the right of the momentum conservation equation (22)). The momentum flux is (using (20))

$$\begin{aligned} \rho v^2 + p(\rho) &= (\rho_0 + \epsilon\tilde{\rho})\epsilon^2\tilde{v} + p(\rho_0) + \epsilon c^2(\rho)\tilde{\rho} + O(\epsilon^2) \\ &= p_0 + \epsilon c_0^2\tilde{\rho} + O(\epsilon^2) . \end{aligned}$$

Here  $p_0 = p(\rho_0)$  and  $c_0 = c(\rho_0)$ . We substitute this into the conservation laws. The constant terms drop. We keep the  $O(\epsilon)$  terms:

$$\partial_t\tilde{\rho} + \rho_0\partial_x\tilde{v} = 0 \tag{23}$$

$$\rho_0\partial_t\tilde{v} + c_0^2\partial_x\tilde{\rho} = 0 . \tag{24}$$

These are the equations of linear acoustics, for one dimensional sound propagation.

### Propagation modes

The linear acoustics equations (23) and (24) have solutions that look like solutions of the constant velocity advection equation. They move with speed  $s$  without changing shape. Unlike advection, there are two propagation speeds,  $s_{\pm}$ . There are two distinct propagation modes. It turns out that  $s_{\pm} = \pm c_0$ . Thus,  $c_0$  is the speed of sound, and sound may propagate at speed  $c_0$  either in the positive or negative directions.

There are several ways to do this kind of analysis. Here is a way that generalizes to many more complicated wave propagation systems. We write the pair of equations in matrix/vector form. We define the vector solution

$$u(x, t) = \begin{pmatrix} \tilde{\rho}(x, t) \\ \tilde{v}(x, t) \end{pmatrix} .$$

You can check that the pair of equations (23) and (24) may be written as

$$\partial_t u + \begin{pmatrix} 0 & \rho_0 \\ \frac{c_0^2}{\rho_0} & 0 \end{pmatrix} \partial_x u = 0 .$$

This equation is a special case of a general first order system of equations

$$\partial_t u + A\partial_x u = 0 . \tag{25}$$

A propagating mode solution with speed  $s$  has a shape function  $\phi$  and a right eigenvector  $r$ . The solution, which builds in the propagation speed  $s$  and the fact that the shape doesn't change, is

$$u(x, t) = \phi(x - st)r . \quad (26)$$

We write  $\phi'$  for the derivative of  $\phi$  with respect to its argument. We plug the wave ansatz (26) into the general system of equations (25) and get

$$\phi'(x - st) - sAr\phi'(x - st) = 0 .$$

If  $\phi$  is not constant, this is possible only if  $r$  is a right eigenvector of  $A$  with eigenvalue  $s$

$$Ar = sr .$$

The system of equations (25) is called *strongly hyperbolic* if  $A$  has a full set of linearly independent eigenvectors with corresponding real eigenvalues. If any of the eigenvalues of  $A$  is not real, or if  $A$  has non-trivial Jordan block structure, then the PDE (25) is “problematic”.

Fortunately, this is not the case for linear acoustics. The eigenvalue problem is

$$\begin{pmatrix} 0 & \rho_0 \\ \frac{c_0^2}{\rho_0} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = s \begin{pmatrix} a \\ b \end{pmatrix} .$$

The characteristic polynomial is

$$f(s) = \det \begin{pmatrix} -s & \rho_0 \\ \frac{c_0^2}{\rho_0} - s & 0 \end{pmatrix} = s^2 - c_0^2 .$$

Setting  $f(s) = 0$  to find eigenvalues gives  $s_{\pm} = \pm c_0$  as claimed. The eigenvectors for  $s_{\pm}$  may be found in the form

$$r_{\pm} = \begin{pmatrix} 1 \\ a_{\pm} \end{pmatrix} .$$

The result is

$$r_{\pm} = \begin{pmatrix} 1 \\ \frac{s_{\pm}}{\rho_0} \end{pmatrix} .$$

These are the *traveling wave* modes for linear acoustics.

The general solution to the initial value problem is a sum of two traveling wave solutions, one with speed  $s_+c_0$  moving to the right and the other with speed  $s_- = -c_0$  moving left. The initial data are

$$u(x, 0) = \begin{pmatrix} \rho(x, 0) \\ v(x, 0) \end{pmatrix} .$$

For any  $x$ , we can expand  $u$  as a sum of the eigenvectors  $r_{\pm}$ . We call the expansion coefficients  $\phi_{\pm}$ :

$$u(x, 0) = \phi_+(x)r_+ + \phi_-(x)r_- .$$

The corresponding solution is

$$u(x, t) = \phi_+(x - c_0 t)r_+ + \phi_-(x + c_0 t)r_- .$$

From this, we can observe some properties of the general solution:

- **Domain of dependence and influence.** The solution at  $(x, t)$  is determined by the initial data at  $x - c_0 t$  and  $x + c_0 t$ . If these are zero, for example, the solution at  $(x, t)$  is zero. If the initial data is zero for  $|x| > L$ , then the solution is zero for  $|x| > L + c_0 t$ .
- **No smoothing.** If the initial data has discontinuities, then the solution keeps them. Discontinuities propagate along characteristic lines.